

# Conditional measure on random sets such as Brownian path and convergence of random measures

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## 1 Introduction

Let  $\nu$  be a finite Borel probability measure on the unit square in the plane, let  $\pi(x, y) = x$  be the projection onto the  $x$ -axis and  $\pi^*\nu = \nu \circ \pi^{-1}$  be the projection measure. Then by the existence of the regular conditional measure [9] for  $\pi^*\nu$  almost every  $x \in [0, 1]$  there exists a Borel probability measure  $\nu_x$  on  $\{x\} \times [0, 1]$  such that  $d\nu(z) = d\nu_x(z)d\pi^*\nu(x)$ . When  $\pi^*\nu \ll \lambda$ , where  $\lambda$  denotes the 1-dimensional Lebesgue measure, the measure  $\nu_x$  can be obtained as a weak limit of certain rescaled restrictions of  $\nu$ . Assume that  $\pi^*\nu \ll \lambda$ , then for Lebesgue almost every  $x \in [0, 1]$  the weak limit of the measures

$$\frac{\nu|_{\pi^{-1}B(x,r)}}{2r} \quad (1)$$

exists as  $r$  approaches 0, see [10, Chapter 10], where  $\nu|_A$  denotes the restriction of  $\nu$  to  $A$ . Let this weak limit be  $\mu_x$  for Lebesgue almost every  $x \in [0, 1]$  then

$$d\nu(z) = d\mu_x(z)d\lambda(x),$$

see Mattila [11, Lemma 3.4]. Thus by the uniqueness of the conditional measure

$$\nu_x = \left( \frac{d\pi^*\nu}{d\lambda}(x) \right)^{-1} \cdot \mu_x$$

for  $\pi^*\nu$  almost every  $x \in [0, 1]$ .

It was shown by Mattila [11, Lemma 3.4] that for a Borel function  $f : [0, 1]^2 \rightarrow \mathbb{R}$  with  $\int |f| d\nu(z) < \infty$  we have that

$$\int f d\mu_x(z) = \lim_{r \rightarrow 0} (2r)^{-1} \int_{B(x,r)} f d\nu(z) \quad (2)$$

for Lebesgue almost every  $x \in [0, 1]$ . Mattila [10, Theorem 10.7] also discusses the double integral of certain kernels with respect to  $\mu_x \times \mu_x$ .

One can look at it as we randomly choose a slice  $B = \{x\} \times [0, 1]$  where we choose  $x$  uniformly in  $[0, 1]$ . Then  $\mu = \mu_x$  is a random measure supported on the random slice, and

$$d\nu(z) = d\mu(z)dP(x)$$

holds. Let  $Q_k(z)$  be the dyadic cube  $[\frac{i_1}{2^{-k}}, \frac{i_1+1}{2^{-k}}) \times [\frac{i_2}{2^{-k}}, \frac{i_2+1}{2^{-k}})$  for  $i_1, i_2 \in \mathbb{Z}$  such that  $z \in Q_k(z)$  for some  $z \in [0, 1]^2$  and let  $\mathcal{Q}_k = \{Q_k(z) : z \in [0, 1]^2\}$ . It can be shown that for Lebesgue almost every  $x \in [0, 1]$  we get the same weak limit  $\mu_x$  if we consider the sequence

$$\mu_k = \frac{\nu|_{\pi^{-1}(\pi(Q_k(x,0)))}}{2^{-k}} = \sum_{Q \in \mathcal{Q}_k} P(Q \cap B \neq \emptyset)^{-1} \cdot I_{Q \cap B \neq \emptyset} \cdot \nu|_Q \quad (3)$$

instead of (1), where  $I_{Q \cap B \neq \emptyset}$  is the indicator function of the event  $Q \cap B \neq \emptyset$ . We can obtain from the analogue of (2) that

$$\lim_{k \rightarrow \infty} \mu_k(A) = \mu(A) \quad (4)$$

almost surely.

If  $\pi^*\nu$  is singular to the Lebesgue measure than by the Lebesgue's density theorem

$$\lim_{k \rightarrow \infty} \mu_k([0, 1]^2) = 0 \quad (5)$$

for Lebesgue almost every  $x \in [0, 1]$ , i.e.  $\mu = 0$  almost surely. Hence in general we can decompose  $\nu$  into two parts

$$\nu = \nu_R + \nu_\perp \quad (6)$$

such that  $\pi^*\nu_R \ll \lambda$  and  $\pi^*\nu_\perp \perp \lambda$ . Thus almost surely we can determine the weak limit  $\mu$  of the sequence of random measures  $\mu_k$  by (3), (4) and (5). It follows that

$$d\nu_R(z) = d\mu(z)dP. \quad (7)$$

We would like to obtain similar results for more general random sets  $B$  in metric spaces. Our main goal is to show the existence of the limit in the case when  $B$  is a Brownian path or a percolation set.

**Theorem 1.1.** *Let  $B$  be a Brownian path in  $\mathbb{R}^d$  for  $d \geq 3$  and let  $\nu$  be a locally finite Borel measure on  $\mathbb{R}^d$ . Then  $\nu = \nu_R + \nu_\perp$  such that there exists  $A$  such that  $\nu_\perp(\mathbb{R}^d \setminus A) = 0$ ,  $P(B \cap A \neq \emptyset) = 0$  and there exists a random locally finite Borel measure  $\mu$  supported on  $B$  such that*

$$d\mu(x)dP = d\nu_R(x)$$

and

$$d\mu(y)d\mu(x)dP = \frac{\|x\|^{d-2} + \|y\|^{d-2}}{\|x - y\|^{d-2}} d\nu_R(x)d\nu_R(y).$$

We use the sum in (3) to define the sequence  $\mu_k$  for more general random sets (see Section (2)). In Section (4) we introduce the concept of weak convergence of random measures in probability and the vague convergence of random measures in probability. In Section we obtain a certain decomposition of the measure  $\nu$  into two parts, similar in nature to (6), such that one part corresponds to a vanishing limit (see Section (6)), the other part corresponds to a regular full action (see Section (8)). In Section (9) we show that for a family of random sets  $B$  we have that  $\mu_k$  converges vaguely in probability to a random measure  $\mu$  for which (7) holds. In Section (10) we discuss the double integration with respect to  $\mu \times \mu$ . In Section (11) we provide estimates of the probability of  $\mu$  being 0.

## 2 Notations

Let  $(X, d)$  be a complete, separable metric space. Let  $\varphi : [0, \infty) \rightarrow [0, \infty]$  be a continuous monotone decreasing function with finite values on  $(0, \infty)$ . Unless stated otherwise, we assume that for small enough  $\delta > 0$  there exist  $c_2, c_3 < \infty$  such that

$$\varphi(r) \leq c_2 \varphi(r \cdot (1 + 2\delta)) + c_3 \quad (8)$$

for all  $r > 0$ . We consider the composition kernel  $\varphi(d(x, y))$  on  $X \times X$  which we denote by  $\varphi(x, y)$ . We note that we use  $\varphi$  to denote both  $\varphi(r)$  and  $\varphi(x, y)$  but in the context it should be clear depending on what is the domain of  $\varphi$ .

Let  $\mathcal{Q}_k$  be a sequence of countable families of Borel subsets of  $X$  such that  $Q \cap S = \emptyset$  for  $Q, S \in \mathcal{Q}_k$ , for all  $k \in \mathbb{N}$  and

$$\lim_{k \rightarrow \infty} \sup \{diam(Q) : Q \in \mathcal{Q}_k\} = 0, \quad (9)$$

where  $diam$  denotes the diameter in  $X$ . We further assume that for every  $Q \in \mathcal{Q}_k$ ,  $k > 1$  there exists a unique  $D \in \mathcal{Q}_{k-1}$  such that

$$Q \subseteq D. \quad (10)$$

Define  $X_0 := \bigcap_{k=1}^{\infty} (\bigcup \mathcal{Q}_k)$ .

Let  $\nu$  be a finite Borel measure in  $X$  such that  $\nu(\{x\}) = 0$  for every  $x \in X$ . We usually assume that  $\nu(X \setminus X_0) = 0$ .

Let  $(\Omega, \mathcal{A}, P)$  be a probability space and  $B = B_\omega \subseteq X$  be a random set such that  $\{B \cap K \neq \emptyset\} \in \mathcal{A}$  for every Borel set  $K \subseteq X$ . We assume that

$$P(Q \cap B \neq \emptyset) > 0 \quad (11)$$

for every  $Q \in \mathcal{Q}_k$ .

We write  $\mu_k = \mathcal{C}_k(\nu) = \sum_{Q \in \mathcal{Q}_k} P(Q \cap B \neq \emptyset)^{-1} \cdot I_{Q \cap B \neq \emptyset} \cdot \nu|_Q$ . It follows that

$$E(\mu_k(A)) = \nu(A) \quad (12)$$

for every Borel set  $A \subseteq X$ .

For our main results we assume that  $\mathcal{Q}_k$  and  $B$  satisfies the following properties. There exists  $\delta > 0$  and  $M_\delta < \infty$  independent of  $k$  such that for every  $Q \in \mathcal{Q}_k$

$$\#\{S \in \mathcal{Q}_k : \max\{diam(Q), diam(S)\} \geq \delta \cdot dist(Q, S)\} \leq M_\delta, \quad (13)$$

where  $dist(Q, S) = \inf_{x \in Q, y \in S} \|x - y\|$ . There exists  $0 < M < \infty$ , independent of  $k$ , such that

$$0 < diam(Q)/M \leq diam(S) \leq diam(Q) \cdot M < \infty \quad (14)$$

for every  $Q, S \in \mathcal{Q}_k$ .

The  $\varphi$ -energy of a Borel measure  $\nu$  on  $X$  is  $I_\varphi(\nu) = \int \int \varphi(x, y) d\nu(x) d\nu(y)$ . The  $\varphi$ -capacity of a Borel subset  $K \subseteq X$  is

$$C_\varphi(K) = \sup \{I_\varphi(\nu)^{-1} : \nu \text{ is a Borel probability measure on } K\}.$$

For our main results we assume that  $\varphi$ ,  $\mathcal{Q}_k$  and  $B$  satisfies the following properties. There exists  $a > 0$  such that

$$aC_\varphi(Q) \leq P(Q \cap B \neq \emptyset) \quad (15)$$

for every  $Q \in \mathcal{Q}_k$ . Note that if  $C_\varphi(Q) > 0$  for every  $Q \in \mathcal{Q}_k$  then (11) holds. There exists  $0 < \delta < 1$  and  $0 < c < \infty$  such that whenever  $Q \in \mathcal{Q}_k$ ,  $D \in \mathcal{Q}_n$  and  $\max \{diam(Q), diam(S)\} < \delta \cdot dist(Q, S)$  then

$$P(Q \cap B \neq \emptyset \text{ and } S \cap B \neq \emptyset) \leq c \cdot P(Q \cap B \neq \emptyset) \cdot P(S \cap B \neq \emptyset) \cdot \varphi(dist(Q, S)). \quad (16)$$

Definialni  $\text{supp}(\mu), dist, B(x, r)$

### 3 Preliminary remarks

For  $x \in X$ ,  $r > 0$  let  $B(x, r) = \{y \in X : d(x, y) < r\}$  and for a set  $A \subseteq X$  let  $B(A, r) = \{y \in X : x \in A, d(x, y) < r\}$ . We denote by  $\mathcal{B}(X)$  the set of Borel subsets of  $X$ . Let  $C_b(X)$  denote the space of bounded continuous functions of  $X$  equipped with the supremum norm and  $C_c(X)$  denotes the space of all compactly supported continuous functions on  $X$  equipped with the supremum norm. For a measure  $\nu$  on  $X$  let  $\text{supp}\nu$  denote the support of  $\nu$ .

*Notation 3.1.* For  $A \subseteq X$  let

$$\chi_A = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

be the characteristic function of  $A$ .

*Notation 3.2.* For a probability event  $A \in \mathcal{A}$  let

$$I_A = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

be the indicator function of  $A$ .

*Remark 3.3.* Let  $Y$  and  $Y_n$  be a sequence of real valued random variables. Then  $Y_n$  converges to  $Y$  in probability if and only if for every subsequence  $k_m$  of  $\mathbb{N}$  we can find a subsequence  $k_{m_n}$  of  $k_m$  such that  $Y_{k_{m_n}}$  converges to  $Y$  almost surely. The convergence in probability is a metric convergence and can be metrizable by the following metric.

*Notation 3.4.* For  $Y, Z$  randomvariables let

$$\rho(Y, Z) = E \left( \frac{|Y - Z|}{1 + |Y - Z|} \right).$$

Note that  $\rho(Y, Z) \leq E(|Y - Z|)$ . We use this fact without reference throughout the paper.

**Lemma 3.5.** *Let  $Y$  be a topological space. Then  $y_n$  converges to  $y$  if and only if for every subsequence  $k_m$  of  $\mathbb{N}$  we can find a subsequence  $k_{m_n}$  of  $k_m$  such that  $y_{k_{m_n}}$  converges to  $y$ .*

**Lemma 3.6.** *Let  $f_n$  be a sequence of random variables such that  $f_n$  converges to  $f$  in probability and  $\lim_{n \rightarrow \infty} E(f_n) = E(f) < \infty$ . Then  $f_n$  converges to  $f$  in  $\mathcal{L}^1$ .*

*Proof.* For every subsequence  $k_m$  of  $\mathbb{N}$  we can find a subsequence  $k_{m_n}$  of  $k_m$  such that  $f_{k_{m_n}}$  converges to  $f$  almost surely. Then  $f_{k_{m_n}}$  converges to  $f$  in  $\mathcal{L}^1$  by Sheffe's Lemma [1]. Since  $\mathcal{L}^1$  convergence is topological convergence it follows by Lemma 3.5 that  $f_n$  converges to  $f$  in  $\mathcal{L}^1$ .  $\square$

**Lemma 3.7.** *Assume that  $\{f_k^i\}_{i,k \in \mathbb{N}}$  is a family of random variables such that  $f_k^i$  converges in probability as  $k$  goes to infinity for every  $i$ . Then for every subsequence  $\{n_k\}_{k=1}^\infty$  of  $\mathbb{N}$  there exists a subsequence  $\{j_k\}_{k=1}^\infty$  of  $\{n_k\}_{k=1}^\infty$  and there exists an event  $H \in \mathcal{A}$  with  $P(H) = 1$  such that  $f_{j_k}^i$  converges on the event  $H$  as  $k$  goes to infinity for every  $i$ .*

*Proof.* Let  $\{\alpha_k^1\}_{k=1}^\infty$  be a subsequence of  $\{n_k\}_{k=1}^\infty$  such that  $f_{\alpha_k^1}^1$  converges almost surely and let  $j_1 = \alpha_1^1$ . If  $\{\alpha_k^i\}_{k=i}^\infty$  and  $j_1, \dots, j_i$  are defined let  $\{\alpha_k^{i+1}\}_{k=i+1}^\infty$  be a subsequence of  $\{\alpha_k^i\}_{k=i+1}^\infty$  be such that  $f_{\alpha_k^{i+1}}^{i+1}$  converges almost surely as  $k$  goes to infinity and let  $j_{i+1} = \alpha_{i+1}^{i+1}$ . Then  $\{j_k\}_{k=i}^\infty$  is a subsequence of  $\{\alpha_k^i\}_{k=i}^\infty$  and hence  $f_{j_k}^i$  converges almost surely as  $k$  goes to infinity for every  $i$ .  $\square$

**Lemma 3.8.** *If  $Y_n$  is a sequence of random variables,  $Y_n$  converges to  $Y$  in probability and  $Y_n < \infty$  almost surely for every  $n \in \mathbb{N}$  then  $Y < \infty$  almost surely.*

*Proof.* If  $Y = \infty$  on an event  $H$  then  $P(H) \leq P(|Y_n - Y| > \varepsilon)$  for every  $n$  and so  $P(H) = 0$ .  $\square$

**Lemma 3.9.** *If  $Y_n$  is a sequence of nonnegative random variables,  $Y_n$  converges to  $Y$  in probability and there exists  $c < \infty$  such that  $E(Y_n) \leq c$  for every  $n \in \mathbb{N}$  then  $E(Y) \leq c$ .*

*Proof.* Let  $n_k$  be a sequence such that  $Y_{n_k}$  converges to  $Y$  almost surely. Then by Fatou's Lemma

$$E(Y) = E(\liminf_{k \rightarrow \infty} Y_{n_k}) \leq \liminf_{k \rightarrow \infty} E(Y_{n_k}) \leq c.$$

$\square$

**Lemma 3.10.** *Let  $\nu$  be a finite Borel measure on  $X$  and  $f : X \rightarrow \mathbb{R}$  be a bounded Borel function. Then for every  $\varepsilon > 0$  there exists  $g \in C_b(X)$  and an open set  $G \subseteq X$  such that  $f(x) = g(x)$  on  $\text{supp } \nu \setminus G$ ,  $\|g\|_\infty \leq \|f\|_\infty$  and  $\nu(G) < \varepsilon$ .*

*Proof.* Since  $X$  is a separable metric space by Luzin's theorem there exists a closed set  $F \subseteq X$  such that  $f|_F$  is continuous and  $\nu(X \setminus F) < \varepsilon$ . By Tietze's extension theorem there exists a continuous extension  $g$  of  $f|_F$  to the whole  $X$  such that  $\|g\|_\infty \leq \|f\|_\infty$ . Then the statement holds for  $G = X \setminus F$ .  $\square$

*Remark 3.11.* In Lemma 3.10 if  $f$  is compactly supported and  $X$  is locally compact then we can further assume that  $\overline{G}$  is compact and  $g \in C_c(X)$ . It is because we can find an open set  $U$  such that  $\text{supp}(f) \subseteq U$ ,  $\overline{U}$  is compact and we can assume that  $F$  contains  $X \setminus U$ . Furthermore, it is enough to assume that  $\nu$  is a locally finite rather than a finite measure.

**Lemma 3.12.** *Let  $\nu$  be a finite, Borel measure on  $X$ . Then there exists a sequence of disjoint compact subsets  $K_1, K_2, \dots$  of  $X$  such that  $\nu(X \setminus \cup_{i=1}^{\infty} K_i) = 0$ .*

*Proof.* By inner regularity [3] we can find  $K_1 \subseteq X$  such that  $\nu(X \setminus K_1) < 1$ . Once we have  $K_1, \dots, K_n$  we can find, by inner regularity,  $K_{n+1} \subseteq X \setminus \cup_{i=1}^n K_i$  such that  $\nu(X \setminus \cup_{i=1}^{n+1} K_i) < 1/n$ . After countably many steps we end up with the desired sequence.  $\square$

*Remark 3.13.*  $\mu_k$  converges to  $\mu$  weakly if and only if  $\mu(G) \leq \liminf_{k \rightarrow \infty} \mu_k(G)$  for every open set  $G$ . See [2, Theorem 2.1].

**Lemma 3.14.** *Let  $\Psi \subseteq C_b(X)$  be a dense subset with respect to the supremum norm. Assume that  $\mu$  and  $\nu$  are finite Borel measures on  $X$  such that  $\int_X f(x) d\mu(x) = \int_X f(x) d\nu(x)$  for every  $f \in \Psi$ . Then  $\mu = \nu$ .*

*Proof.* Let  $f$  be a bounded continuous function. Let  $\varepsilon > 0$  and  $g \in \Psi$  be such that  $\|f - g\|_{\infty} < \varepsilon$ . Then

$$\begin{aligned} & \left| \int_X f(x) d\mu(x) - \int_X f(x) d\nu(x) \right| \\ & \leq \left| \int_X f(x) d\mu(x) - \int_X g(x) d\mu(x) \right| + \left| \int_X g(x) d\nu(x) - \int_X f(x) d\nu(x) \right| \\ & \leq \varepsilon(\mu(X) + \nu(X)). \end{aligned}$$

Since  $\varepsilon > 0$  was arbitrary it follows that  $\int_X f(x) d\mu(x) = \int_X f(x) d\nu(x)$  for every bounded continuous function  $f$ . Hence by Lemma 3.10 and the dominated convergence theorem it follows that  $\mu(A) = \nu(A)$  for every Borel set  $A$ .  $\square$

**Lemma 3.15.** *Assume that  $\mu_k$  converges to both  $\mu$  and  $\nu$  weakly then  $\mu = \nu$ .*

Lemma 3.15 follows from Lemma 3.14.

**Definition 3.16.** The *Prohorov distance* between two finite Borel measures  $\mu$  and  $\nu$  on  $X$  is

$$\pi(\mu, \nu) = \inf \{ \varepsilon : \mu(A) \leq \nu(B(A, \varepsilon) + \varepsilon \text{ and } \nu(A) \leq \mu(B(A, \varepsilon) + \varepsilon \text{ for } \forall A \in \mathcal{B}(X) \}$$

**Lemma 3.17.** *We have that*

$$\pi(\mu, \nu) \leq \mu(X) + \nu(X).$$

The statement of Lemma 3.17 follows from the definition of Prohorov distance.

**Lemma 3.18.** *We have that*

$$\pi(\mu + \nu, \nu) \leq \mu(X).$$

The statement of Lemma 3.17 and Lemma (3.18) follow from the definition of Prohorov distance.

**Proposition 3.19.** *The Prohorov distance is a metric and if  $X$  is separable then  $\mu_k$  weakly converges to  $\mu$  if and only if  $\lim_{k \rightarrow \infty} \pi(\mu_k, \mu) = 0$ .*

For the proof see [2, page 72]

**Lemma 3.20.** *If  $K \subseteq X$  is compact then  $C_b(K)$  is separable.*

See [12, page 437].

## 4 Convergence of random measures

**Definition 4.1.** The set of all finite Borel measures  $\mathcal{M}_+(X)$  on  $X$  equipped with the weak\*-topology on the dual space of  $C_B(X)$  is a Polish space. A random, finite, Borel measure is an element of  $\mathcal{L}^0(\mathcal{M}_+(X))$ , i.e. a finite Borel measure valued random variable.

**Lemma 4.2.** *Let  $\mu_k$  be a sequence of random, finite Borel measures. If there exists  $H \in \mathcal{A}$  with  $P(H) = 1$  such that for every outcome  $\omega \in H$  we have that  $\mu_k$  weakly converges to a finite, Borel measure  $\mu$  (note that  $\mu$  depends on  $\omega \in H$ ) then  $\mu$  is a random, finite Borel measure.*

The lemma follows by the fact that in Polish spaces the limit of random variables is a random variable.

### 4.1 Weak convergence subsequentially in probability

**Definition 4.3.** Let  $\mu$  and  $\mu_k$  be a sequence of random, finite, Borel measures on  $X$ . We say that  $\mu_k$  weakly converges to  $\mu$  subsequentially in probability if for every subsequence  $\{\alpha_k\}_{k=1}^\infty$  of  $\mathbb{N}$  there exists a subsequence  $\{\beta_k\}_{k=1}^\infty$  of  $\{\alpha_k\}_{k=1}^\infty$  and an event  $H \in \mathcal{A}$  with  $P(H) = 1$  such that  $\mu_{\beta_k}$  converges weakly to  $\mu$  on the event  $H$ .

*Remark 4.4.* It follows from Definition 4.21 that if  $\mu_k$  is a sequence of random, finite, Borel measures on  $X$  such that  $\mu_k$  weakly converges to a random Borel measure  $\mu$  almost surely then  $\mu_k$  weakly converges to  $\mu$  subsequentially in probability.

**Proposition 4.5.** *The limit in Definition 4.3 is unique in  $\mathcal{L}^0(\mathcal{M}_+(X))$  if exists.*

*Proof.* Assume that a random sequence of measures converges weakly in probability to both of the random measures  $\mu$  and  $\nu$ . Then there exists  $\{\alpha_k\}_{k=1}^\infty$  and an event  $H \in \mathcal{A}$  with  $P(H) = 1$  such that  $\mu_{\alpha_k}$  weakly converges to both  $\mu$  and  $\nu$  on the event  $H$ . Hence  $\mu = \nu$  by Lemma 3.15.  $\square$

**Definition 4.6.** Let  $\mu$  and  $\nu$  be two random, finite, Borel measures. We define

$$\rho_\pi(\mu, \nu) = E \left( \frac{\pi(\mu, \nu)}{1 + \pi(\mu, \nu)} \right).$$

**Proposition 4.7.** *We have that  $\rho_\pi$  is a metric and  $\mu_k$  weakly converges to  $\mu$  subsequentially in probability if and only if  $\lim_{k \rightarrow \infty} \rho_\pi(\mu_k, \mu) = 0$ .*

*Proof.* The fact that  $\rho_\pi$  is a metric can be shown similarly that  $\rho$  is a metric, depending on the fact that  $x/1+x$  is monotone increasing convex function on the positive reals. The statement follows from Proposition 3.19 and Remark 3.3.  $\square$

**Proposition 4.8.** *Let  $\mu_k$  be a sequence of random, finite, Borel measures on  $X$ . If  $\mu_k$  weakly converges to a random, finite, Borel measure  $\mu$  subsequentially in probability then  $\int_X f(x)d\mu_k(x)$  converges to  $\int_X f(x)d\mu(x)$  in probability for every  $f \in C_b(X)$ .*

*Proof.* Let  $f \in C_b(X)$ . For every subsequence  $\{\alpha_k\}_{k=1}^\infty$  of  $\mathbb{N}$  there exists a subsequence  $\{\beta_k\}_{k=1}^\infty$  of  $\{\alpha_k\}_{k=1}^\infty$  and an event  $H \in \mathcal{A}$  with  $P(H) = 1$  such that  $\mu_{\beta_k}$  converges weakly to  $\mu$  on the event  $H$ . Thus  $\int_X f(x)d\mu_{\beta_k}(x)$  converges to  $\int_X f(x)d\mu(x)$  almost surely. Hence  $\int_X f(x)d\mu_k(x)$  converges to  $\int_X f(x)d\mu(x)$  in probability by Remark 3.3.  $\square$

**Lemma 4.9.** *Let  $K \subseteq X$  be a compact subset, let  $\Psi \subseteq C_b(K)$  be a dense subset with respect to the supremum norm and let  $\mu_k$  be a sequence of deterministic, finite, Borel measures on  $K$  such that  $\int_X f(x)d\mu_k(x)$  converges to a limit  $S(f) < \infty$  for every  $f \in \Psi$ . Then  $\mu_k(K)$  is bounded and  $\mu_k$  converges weakly to a finite, Borel measure.*

*Proof.* Let  $g \in \Psi$  such that  $\|\chi_K - g\| < 1/2$ . Then  $\chi_K \leq 2g$  on  $K$  thus  $\limsup \mu_k(K) \leq 2 \int_X g(x)d\mu(x) = S(g) < \infty$  and so  $\mu_k(K)$  is bounded.

Since  $\mu_k(K)$  is bounded and  $K$  is a compact metric space it follows that there exists a subsequence  $n_k$  of  $\mathbb{N}$  such that  $\mu_{n_k}$  weakly converges to a Borel measure  $\tau$  of finite total mass. Let  $f$  be a bounded continuous function and  $g \in \Psi$  be such that  $\|f - g\|_\infty < \varepsilon$ . Then

$$\begin{aligned} & \limsup_{k \rightarrow \infty} \left| \int_X f(x)d\tau(x) - \int_X f(x)d\mu_k(x) \right| \\ & \leq \limsup_{k \rightarrow \infty} \left| \int_X f(x)d\tau(x) - \int_X g(x)d\tau(x) \right| + \left| \int_X g(x)d\tau(x) - \int_X g(x)d\mu_k(x) \right| + \left| \int_X f(x)d\mu_k(x) - \int_X g(x)d\mu_k(x) \right| \\ & \leq \int_X |f(x) - g(x)| d\tau(x) + \limsup_{k \rightarrow \infty} \int_X |f(x) - g(x)| d\mu_k(x) \leq \varepsilon \left( \tau(K) + \limsup_{k \rightarrow \infty} \mu_k(X) \right). \end{aligned}$$

By taking  $\varepsilon$  goes to 0 it follows that  $\int_X f(x)d\tau(x) = \lim_{k \rightarrow \infty} \int_X f(x)d\mu_k(x)$ .  $\square$

**Proposition 4.10.** *Let  $K \subseteq X$  be a compact subset, let  $\Psi \subseteq C_b(K)$  be a countable dense subset with respect to the supremum norm and let  $\mu$  and  $\mu_k$  be a sequence of random, finite Borel measures on  $K$ . If  $\int_X f(x)d\mu_k(x)$  converges to  $\int_X f(x)d\mu(x)$  in probability for every  $f \in \Psi$  then  $\mu_k$  weakly converges to  $\mu$  subsequentially in probability.*

*Proof.* By Lemma 3.7 for every subsequence  $\{\alpha_k\}_{k=1}^\infty$  of  $\mathbb{N}$  there exists a subsequence  $\{\beta_k\}_{k=1}^\infty$  of  $\{\alpha_k\}_{k=1}^\infty$  and there exists an event  $H \in \mathcal{A}$  with  $P(H) = 1$  such that  $\int_X f(x)d\mu_{\beta_k}(x)$  converges to  $\int_X f(x)d\mu(x)$  for every  $f \in \Psi$  on the event  $H$ . We have that  $\mu_{\beta_k}$  weakly converges to a measure  $\tau$  on the event  $H$  by Lemma 4.9. We have that  $\int_X f(x)d\mu(x) = \int_X f(x)d\tau(x)$  for every  $f \in C_b(K)$  on the event  $H$  and hence  $\tau = \mu$  on the event  $H$  by Lemma 3.14. So  $\mu_{\beta_k}$  weakly converges to the measure  $\mu$  on the event  $H$ .  $\square$



**Theorem 4.11.** *Let  $K \subseteq X$  be a compact subset, let  $\Psi \subseteq C_b(K)$  be a countable dense subset with respect to the supremum norm and let  $\mu_k$  be a sequence of random Borel measures on  $K$  such that  $\int_X f(x) d\mu_k(x)$  converges in probability to a random limit  $S(f)$  for every  $f \in \Psi$  and  $S(f) < \infty$  almost surely. Then  $\mu_k$  weakly converges to a random, finite, Borel measure  $\mu$  subsequentially in probability.*

*Proof.* By Lemma 3.7 there exists a subsequence  $\{\beta_k\}_{k=1}^\infty$  of  $\mathbb{N}$  and there exists an event  $H \in \mathcal{A}$  with  $P(H) = 1$  such that  $\int_X f(x) d\mu_{\beta_k}(x)$  converges to  $S(f)$  for every  $f \in \Psi$  on the event  $H$ . Then  $\mu_{\beta_k}$  weakly converges to a random, finite, Borel measure  $\mu$  on the event  $H$  by Lemma 4.9 and Lemma 4.2. Thus  $\int_X f(x) d\mu_k(x)$  converges to  $S(f) = \int_X f(x) d\mu(x)$  in probability for every  $f \in \Psi$  and so  $\mu_k$  weakly converges to  $\mu$  in probability by Proposition 4.10.  $\square$

**Proposition 4.12.** *Let  $\nu$  be a deterministic Borel measure on  $X$  and  $\mu_k$  be a sequence of random, finite, Borel measures on  $X$  such that  $\mu_k \ll \nu$  almost surely for every  $k$ , there exists  $c > 0$  such that  $E\left(\frac{d\mu_k(x)}{d\nu(x)}\right) \leq c$  for every  $k$  and  $\mu_k(A)$  converges in probability for every Borel set  $A \subseteq X$ . Let  $f : X \rightarrow \mathbb{R}$  be a Borel measurable function such that  $\int_X |f(x)| d\nu(x) < \infty$ . Then  $\int_X f(x) d\mu_k(x)$  converges to a random variable  $Y$  in probability and  $E(|Y|) \leq c \int_X |f(x)| d\nu(x)$ .*

*Proof.* It is enough to prove the statement for a nonnegative  $f$ . We have that  $E\left(\int_X g(x) d\mu_k(x)\right) = E\left(\int_X g(x) \frac{d\mu_k(x)}{d\nu(x)} d\nu(x)\right) \leq c \int_X g(x) d\nu(x)$  for every nonnegative Borel function  $g$  by Fubini's theorem. Let  $g_n(x) = \sum_{i=1}^{N_n} b_{i,n} \cdot \chi_{A_{i,n}}(x)$ , where  $0 \leq b_i < \infty$ ,  $N_n \in \mathbb{N}$  and  $A_{i,n} \subseteq X$  are Borel subsets, such that  $g_n \leq f$  on  $X$  and  $0 \leq \int_X f(x) - g_n(x) d\nu(x) < 1/n$ . By assumption  $\int_X g_n(x) d\mu_k(x)$  converges in probability as  $k$  goes to infinity to a random variable  $Y_n$ , thus

$$\lim_{k \rightarrow \infty} \rho\left(\int_X g_n(x) d\mu_k(x), Y_n\right) = 0.$$

Hence

$$\begin{aligned} \limsup_{k \rightarrow \infty} \rho\left(\int_X f(x) d\mu_k(x), Y_n\right) &\leq \limsup_{k \rightarrow \infty} \rho\left(\int_X f(x) d\mu_k(x), \int_X g_n(x) d\mu_k(x)\right) + \rho\left(\int_X g_n(x) d\mu_k(x), Y_n\right) \\ &\leq \limsup_{k \rightarrow \infty} E\left|\int_X f(x) d\mu_k(x) - \int_X g_n(x) d\mu_k(x)\right| + 0 \leq \limsup_{k \rightarrow \infty} E\left(\int_X |f(x) - g_n(x)| d\mu_k(x)\right) \\ &\leq c \int_X f(x) - g_n(x) d\nu(x) = c \int_X f(x) - g_n(x) d\nu(x) < c/n, \end{aligned}$$

and so there exists  $m_n \in \mathbb{N}$  such that  $\rho\left(\int_X f(x) d\mu_k(x), Y_n\right) < c/n$  for every  $k \geq m_n$ . If  $k \geq \max\{m_n, m_l\}$  then

$$\rho(Y_n, Y_l) \leq \rho\left(\int_X f(x) d\mu_k(x), Y_n\right) + \rho\left(\int_X f(x) d\mu_k(x), Y_l\right) \leq c/n + c/l.$$

It follows that  $Y_n$  is a Cauchy sequence for the metric  $\rho$ , which is a complete metric. Let  $Y$  be the limit of  $Y_n$  in probability. Then

$$\limsup_{k \rightarrow \infty} \rho \left( \int_X f(x) d\mu_k(x), Y \right) \leq \limsup_{k \rightarrow \infty} \rho \left( \int_X f(x) d\mu_k(x), Y_n \right) + \rho(Y_n, Y) \leq c/n + \rho(Y_n, Y).$$

By taking limit  $n$  goes to infinity it follows that  $\int_X f(x) d\mu_k(x)$  converges to  $Y$  in probability. By applying Lemma 3.9 twice

$$E(Y) \leq \liminf_{n \rightarrow \infty} E(Y_n) \leq c \liminf_{n \rightarrow \infty} \int_X g_n(x) d\nu(x) \leq c \int_X f(x) d\nu(x).$$

□

**Corollary 4.13.** *Let  $\nu$  be a deterministic Borel measure on  $X$  such that  $\text{supp}\nu$  is compact. Let  $\mu_k$  be a sequence of random, finite, Borel measures on  $X$  such that  $\mu_k \ll \nu$  almost surely for every  $k$ , there exists  $c > 0$  such that  $E\left(\frac{d\mu_k(x)}{d\nu(x)}\right) \leq c$  for every  $k$  and  $\mu_k(A)$  converges to a random variable  $\tau(A)$  in probability for every Borel set  $A \subseteq X$ . Then  $\mu_k$  weakly converges to a random, finite, Borel measure  $\mu$  subsequentially in probability.*

*Proof.* There exists  $\Psi \subseteq C_b(\text{supp}\nu)$  countable and dense subset by Lemma 3.20. The conditions of Theorem 4.11 are satisfied for  $K = \text{supp}\nu$  by Proposition 4.12. Thus there exists a random Borel measure  $\mu$  such that  $\mu_k$  weakly converges to  $\mu$  subsequentially in probability. □

**Lemma 4.14.** *Let  $g : X \rightarrow \mathbb{R}$  be a bounded Borel measurable function and  $G \subseteq X$  be an open set such that  $g(x) = 0$  for every  $x \notin G$ . If  $\mu_k$  is a sequence of random, finite, Borel measures such that it weakly converges to a random, finite, Borel measure  $\mu$  subsequentially in probability then  $E\left(\int_X |g(x)| d\mu(x)\right) \leq \|g\|_\infty \cdot \liminf_{k \rightarrow \infty} E(\mu_k(G))$ .*

*Proof.* Let  $\{\alpha_k\}_{k=1}^\infty$  be a subsequence of  $\mathbb{N}$  such that  $\liminf_{k \rightarrow \infty} E(\mu_k(G)) = \lim_{k \rightarrow \infty} E(\mu_{\alpha_k}(G))$ . Then there exists a subsequence  $\{\beta_k\}_{k=1}^\infty$  of  $\{\alpha_k\}_{k=1}^\infty$  and an event  $H \in \mathcal{A}$  with  $P(H) = 1$  such that  $\mu_{\beta_k}$  weakly converges to  $\mu$  on the event  $H$ . Then by Remark 3.13 and Fatou's lemma

$$\begin{aligned} E\left(\int_X |g(x)| d\mu(x)\right) &\leq \|g\|_\infty \cdot E(\mu(G)) \leq \|g\|_\infty \cdot E(\liminf_{k \rightarrow \infty} \mu_{\beta_k}(G)) \\ &\leq \|g\|_\infty \cdot \lim_{k \rightarrow \infty} E(\mu_{\beta_k}(G)) = \|g\|_\infty \cdot \liminf_{k \rightarrow \infty} E(\mu_k(G)). \end{aligned}$$

□

**Lemma 4.15.** *Let  $g : X \rightarrow \mathbb{R}$  be a bounded Borel function, let  $G \subseteq X$  be an open set such that  $g(x) = 0$  for every  $x \notin G$  and let  $\nu$  be a deterministic Borel measure on  $X$ . Let  $\mu_k$  be a sequence of random, finite, Borel measures on  $X$  such that  $\mu_k \ll \nu$  almost surely for every  $k$  and there exists  $c > 0$  such that  $E\left(\frac{d\mu_k(x)}{d\nu(x)}\right) \leq c$  for every  $k$ . If  $\mu_k$  weakly converges to random, finite, Borel measure  $\mu$  subsequentially in probability then  $E\left(\int_X |g(x)| d\mu(x)\right) \leq c \cdot \|g\|_\infty \cdot \nu(G)$ .*

*Proof.* We have that  $E\left(\int_X \chi_G(x) d\mu_k(x)\right) = E\left(\int_X \chi_G(x) \frac{d\mu_k(x)}{d\nu(x)} d\nu(x)\right) \leq c \int_X \chi_G(x) d\nu(x)$  by Fubini's theorem. Hence the statement follows from Lemma 4.14.  $\square$

**Proposition 4.16.** *Let  $\mu^i$  and  $\mu_k^i$  be a sequence of random, finite, Borel measures on  $X$  for every  $i \in \mathbb{N}$ . Assume that  $\mu_k^i$  weakly converges to  $\mu^i$  subsequentially in probability for every  $i \in \mathbb{N}$  as  $k$  goes to  $\infty$ . Assume that for every  $\varepsilon > 0$  there exist  $N, n_0 \in \mathbb{N}$  such that  $\sum_{i=N}^{\infty} E(\mu_k^i(X)) < \varepsilon$  for every  $k \geq n_0$ . Then there exists  $n_1 \in \mathbb{N}$  such that  $\sum_{i \in \mathbb{N}} \mu_k^i$  is a sequence of random, finite Borel measures, for  $k \geq n_1$ , that weakly converges to the random, finite Borel measure  $\sum_{i \in \mathbb{N}} \mu^i$  subsequentially in probability.*

*Proof.* Since there exists  $N_1, n_1 \in \mathbb{N}$  such that  $\sum_{i=N_1}^{\infty} E(\mu_k^i(X)) \leq 1$  for every  $k \geq n_1$  it follows that  $\sum_{i=N_1}^{\infty} \mu_k^i(X) < \infty$  almost surely for  $k \geq n_1$  and so  $\sum_{i=1}^{\infty} \mu_k^i(X) < \infty$  almost surely for  $k \geq n_1$ . Thus  $\sum_{i=1}^{\infty} \mu_k^i$  is a random, finite, Borel measure. We can find, by Lemma 3.7, a subsequence  $\{\alpha_k\}_{k=1}^{\infty}$  of  $\mathbb{N}$  and an event  $H \in \mathcal{A}$  with  $P(H) = 1$  such that  $\mu_{\alpha_k}^i(X)$  converges to  $\mu^i(X)$  on  $H$  as  $k$  goes to  $\infty$  for every  $i \in \mathbb{N}$ . Then by Fatou's lemma and Fubini's theorem

$$E\left(\sum_{i=N_1}^{\infty} \mu^i(X)\right) = E\left(\sum_{i=N_1}^{\infty} \lim_{k \rightarrow \infty} \mu_{\alpha_k}^i(X)\right) \leq \liminf_{k \rightarrow \infty} \sum_{i=N_1}^{\infty} E(\mu_{\alpha_k}^i(X)) \leq 1. \quad (17)$$

Thus  $\sum_{i=N_1}^{\infty} \mu^i(X) < \infty$  almost surely and so  $\sum_{i=1}^{\infty} \mu^i(X) < \infty$  almost surely. Hence  $\sum_{i=1}^{\infty} \mu^i$  is a random, finite, Borel measure.

Let  $\varepsilon > 0$  be fixed and let  $N, n_0 \in \mathbb{N}$  such that  $\sum_{i=N}^{\infty} E(\mu_k^i(X)) < \varepsilon$  for every  $k \geq n_0$ . Similarly to (17) we have that  $E(\sum_{i=N}^{\infty} \mu^i(X)) \leq \varepsilon$ . Thus

$$\begin{aligned} & \rho_{\pi}\left(\sum_{i=1}^{\infty} \mu_k^i(X), \sum_{i=1}^{\infty} \mu^i(X)\right) \leq \\ & \rho_{\pi}\left(\sum_{i=1}^{\infty} \mu_k^i(X), \sum_{i=1}^{N-1} \mu_k^i(X)\right) + \rho_{\pi}\left(\sum_{i=1}^{N-1} \mu_k^i(X), \sum_{i=1}^{N-1} \mu^i(X)\right) + \rho_{\pi}\left(\sum_{i=1}^{N-1} \mu^i(X), \sum_{i=1}^{\infty} \mu^i(X)\right) \\ & \leq \varepsilon + \rho_{\pi}\left(\sum_{i=1}^{N-1} \mu_k^i(X), \sum_{i=1}^{N-1} \mu^i(X)\right) + \varepsilon \end{aligned}$$

where we used the fact that  $\rho_{\pi}(\mu, \nu) \leq E(\pi(\mu, \nu))$  and Lemma (3.18). Thus

$$\limsup_{k \rightarrow \infty} \rho_{\pi}\left(\sum_{i=1}^{\infty} \mu_k^i(X), \sum_{i=1}^{\infty} \mu^i(X)\right) \leq 2\varepsilon$$

since  $\sum_{i=1}^{N-1} \mu_k^i(X)$  weakly converges to  $\sum_{i=1}^{N-1} \mu^i(X)$  subsequentially in probability. By taking limit  $\varepsilon$  goes to 0 it follows that  $\sum_{i=1}^{\infty} \mu_k^i(X)$  weakly converges to  $\sum_{i=1}^{\infty} \mu^i(X)$  subsequentially in probability.  $\square$

**Theorem 4.17.** *Let  $\nu$  be a deterministic, finite, Borel measure on  $X$ . Let  $\mu_k$  be a sequence of random, finite, Borel measures on  $X$  such that  $\mu_k \ll \nu$  almost surely for every  $k$ , there*

exists  $c > 0$  such that  $E\left(\frac{d\mu_k(x)}{d\nu(x)}\right) \leq c$  for every  $k$  and  $\mu_k(A)$  converges to a random variable  $\tau(A)$  in probability for every Borel set  $A \subseteq X$ . Then  $\mu_k$  weakly converges to a random, finite, Borel measure  $\mu$  subsequentially in probability. Furthermore,  $\int_X f(x) d\mu_k(x)$  converges to a random variable  $S(f)$  in probability with  $E(|S(f)|) \leq c \int_X |f(x)| d\nu(x)$  for every  $f : X \rightarrow \mathbb{R}$  Borel measurable function such that  $\int_X |f(x)| d\nu(x) < \infty$ . For every countable collection of bounded, Borel measurable functions  $f_n : X \rightarrow \mathbb{R}$  we have that  $\int_X f_n(x) d\mu(x) = S(f_n)$  almost surely.

*Proof.* Let  $K_1, K_2, \dots$  be a sequence of disjoint compact subsets of  $X$  by Lemma 3.12. Let  $\nu^i = \nu|_{K_i}$  and  $\mu_k^i = \mu_k|_{K_i}$ . Then  $\mu_k = \sum_{i=1}^{\infty} \mu_k^i$  and  $\mu_k^i$  weakly converges to a random, finite Borel measure  $\mu^i$  subsequentially in probability by Corollary 4.13. Since  $\sum_{i=N}^{\infty} E(\mu_k^i(X)) \leq c \sum_{i=N}^{\infty} \nu(K_i) < \infty$  it follows that the conditions of Proposition 4.16 are satisfied and so  $\mu = \sum_{i=1}^{\infty} \mu^i$  is a random, finite, Borel measure and  $\mu_k$  weakly converges to  $\mu$  subsequentially in probability.

By Proposition 4.12 it follows that  $\int_X f(x) d\mu_k(x)$  converges to a random variable  $S(f)$  in probability with  $E(|S(f)|) \leq c \int_X |f(x)| d\nu(x)$  for every  $f : X \rightarrow \mathbb{R}$  Borel measurable function such that  $\int_X |f(x)| d\nu(x) < \infty$ .

Let  $\varepsilon > 0$ . For a fix  $n \in \mathbb{N}$  we can find  $g \in C_b(X)$  and an open set  $G \subseteq X$  such that  $f_n(x) = g(x)$  on  $X \setminus G$ ,  $\|g\|_{\infty} \leq \|f_n\|_{\infty}$  and  $\nu(G) < \varepsilon$  by Lemma 3.10. Then

$$\begin{aligned}
& \rho\left(\int_X f_n(x) d\mu(x), S(f_n)\right) \\
& \leq \rho\left(\int_X f_n(x) d\mu(x), \int_X g(x) d\mu(x)\right) + \rho\left(\int_X g(x) d\mu(x), \int_X g(x) d\mu_k(x)\right) \\
& \quad + \rho\left(\int_X g(x) d\mu_k(x), \int_X f_n(x) d\mu_k(x)\right) + \rho\left(\int_X f_n(x) d\mu_k(x), S(f_n)\right) \\
& \leq E\left(\int_X |f_n(x) - g(x)| d\mu(x)\right) + \rho\left(\int_X g(x) d\mu(x), \int_X g(x) d\mu_k(x)\right) \\
& \quad + E\left(\int_X |g(x) - f_n(x)| d\mu_k(x)\right) + \rho\left(\int_X f_n(x) d\mu_k(x), S(f_n)\right) \\
& \leq 2c \|f_n\|_{\infty} \cdot \nu(G) + \rho\left(\int_X g(x) d\mu(x), \int_X g(x) d\mu_k(x)\right) + 2c \|f_n\|_{\infty} \cdot \nu(G) + \rho\left(\int_X f_n(x) d\mu_k(x), S(f_n)\right)
\end{aligned}$$

where we used Lemma 4.15 and that  $E(\mu_k(G)) \leq c \cdot \nu(G)$  by Fubini's theorem. Hence taking limit  $k$  goes to infinity it follows that

$$\rho\left(\int_X f_n(x) d\mu(x), S(f_n)\right) \leq 4c \|f_n\|_{\infty} \cdot \nu(G) < 4c \|f\|_{\infty} \cdot \varepsilon$$

since  $\lim_{k \rightarrow \infty} \rho \left( \int_X g(x) d\mu(x), \int_X g(x) d\mu_k(x) \right) = 0$  by Proposition 4.8 and  $\lim_{k \rightarrow \infty} \rho \left( \int_X f_n(x) d\mu_k(x), S(f_n) \right) = 0$  by the definition of  $S(f_n)$ . Taking limit  $\varepsilon$  goes to 0 it follows that  $\int_X f_n(x) d\mu(x) = S(f_n)$  almost surely. Thus we have that  $\int_X f_n(x) d\mu(x) = S(f_n)$  for every  $n \in \mathbb{N}$  almost surely.  $\square$

*Remark 4.18.* In particular, if we take  $f_n = \chi_{A_n}$  in Theorem 4.17 for a countable collection of Borel sets  $A_n$  then it follows that  $\mu(A_n) = S(\chi_{A_n})$  almost surely for every  $n$  where  $\mu_k(A_n)$  converges to  $S(\chi_{A_n})$  in probability as  $k$  goes to infinity.

*Remark 4.19.* In Theorem 4.17 we can relax the condition  $E \left( \frac{d\mu_k(x)}{d\nu(x)} \right) \leq c$  for every  $k$ . It is enough to assume that there exists a nonnegative Borel function  $c : X \rightarrow \mathbb{R}$  such that  $\int c(x) d\nu(x) < \infty$  and  $E \left( \frac{d\mu_k(x)}{d\nu(x)} \right) \leq c(x)$  for  $\nu$  almost every  $x \in X$  for every  $k$ . It can be seen by replacing  $d\nu(x)$  by  $c(x)d\nu(x)$ .

**Proposition 4.20.** *Let  $\mu_k$  be a sequence of random, finite, Borel measures on  $X$  such that there exists a sequence of random closed sets  $F_1 \supseteq F_2 \supseteq \dots$  such that  $\text{supp}\mu_k \subseteq F_k$  almost surely. If  $\mu_k$  weakly converges to a random Borel measure  $\mu$  in probability then  $\text{supp}\mu \subseteq \bigcap_{n=1}^{\infty} F_n$ .*

*Proof.* Let  $G_n = X \setminus F_n$  be a random open set. Let  $\{\alpha_k\}_{k=1}^{\infty}$  be a subsequence of  $\mathbb{N}$  such that  $\mu_{\alpha_k}$  weakly converges to  $\mu$  on the event  $H \in \mathcal{A}$  and  $P(H) = 1$ . Then  $\mu(G_n) \leq \liminf_{k \rightarrow \infty} \mu_{\alpha_k}(G_n) = 0$  on the event  $H$  by Remark 3.13. Hence  $\mu \left( \bigcup_{n=1}^{\infty} G_n \right) = 0$  on the event  $H$  and so  $\text{supp}\mu \subseteq \bigcap_{n=1}^{\infty} F_n$  almost surely.  $\square$

## 4.2 Weak convergence in probability

**Definition 4.21.** Let  $\mu_k$  be a sequence of random, finite, Borel measures on  $X$ . We say that  $\mu_k$  *weakly converges to a random, finite, Borel measure  $\mu$  in probability* if  $\int_X f(x) d\mu_k(x)$  converges to  $\int_X f(x) d\mu(x)$  in probability for every deterministic, bounded, continuous function  $f : X \rightarrow \mathbb{R}$ .

**Proposition 4.22.** *The convergence weakly in probability is induced by the topology which has base elements formed by finite intersection of sets in the form:*

$$\left\{ \mu : \rho \left( \int_X f(x) d\mu(x), Y \right) < r \right\}$$

where  $f \in C_b(X)$ ,  $r > 0$  and  $Y$  is a real-valued random variable with almost surely finite values.

**Proposition 4.23.** *Let  $\mu$  and  $\mu_k$  be a sequence of random, finite, Borel measures on  $X$ . If  $\mu_k$  weakly converges to  $\mu$  subsequentially in probability then  $\mu_k$  weakly converges to  $\mu$  in probability.*

Proposition 4.23 is a reformulation of Proposition 4.8.

**Proposition 4.24.** *Let  $\mu$  and  $\mu_k$  be a sequence of random, finite, Borel measures on a deterministic compact subset  $K \subseteq X$ . If  $\mu_k$  weakly converges to  $\mu$  in probability then  $\mu_k$  weakly converges to  $\mu$  subsequentially in probability.*

*Proof.* We can find a countable and dense  $\Psi \subseteq C_b(K)$  by Lemma 3.20. Hence the statement follows from Proposition 4.10.  $\square$

**Proposition 4.25.** *Let  $\mu^i$  and  $\mu_k^i$  be a sequence of random, finite, Borel measures on  $X$  for every  $i \in \mathbb{N}$ . Assume that  $\mu_k^i$  weakly converges to  $\mu^i$  in probability for every  $i \in \mathbb{N}$  as  $k$  goes to  $\infty$ . Assume that for every  $\varepsilon > 0$  there exist  $N, n_0 \in \mathbb{N}$  such that  $\sum_{i=N}^{\infty} E(\mu_k^i(X)) < \varepsilon$  for every  $k \geq n_0$ . Then there exists  $n_1 \in \mathbb{N}$  such that  $\sum_{i \in \mathbb{N}} \mu_k^i$  is a sequence of random, finite Borel measures, for  $k \geq n_1$ , that weakly converges to the random, finite Borel measure  $\sum_{i \in \mathbb{N}} \mu^i$  in probability.*

The proof of Proposition (4.25) goes similarly to the proof of Proposition (4.16) with the difference that instead of  $\rho_{\pi}(\sum_{i=1}^{\infty} \mu_k^i(X), \sum_{i=1}^{\infty} \mu^i(X))$  we need to estimate  $\rho(\sum_{i=1}^{\infty} \int_X f(x) d\mu_k^i(x), \sum_{i=1}^{\infty} \int_X f(x) d\mu^i(x))$  for a given  $f \in C_b(X)$ . We leave for the reader to check the details. We provide a similar proof to Proposition (4.36).

### 4.3 Vague convergence subsequentially in probability

**Definition 4.26.** The set of all locally finite Borel measures  $\mathcal{M}_l(X)$  on  $X$  equipped with the weak\*-topology on the dual space of  $C_c(X)$  is a Polish space. A random, finite, Borel measure is an element of  $\mathcal{L}^0(\mathcal{M}_l(X))$ , i.e. a locally finite Borel measure valued random variable.

**Definition 4.27.** Let  $\mu$  and  $\mu_k$  be a sequence of random, locally finite, Borel measures on  $X$ . We say that  $\mu_k$  vaguely converges to  $\mu$  in probability if  $\int_X f(x) d\mu_k(x)$  converges to  $\int_X f(x) d\mu(x)$  in probability for every  $f \in C_c(X)$ .

*Remark 4.28.* It follows from Definition 4.21 that if  $\mu_k$  is a sequence of random, locally finite, Borel measures on  $X$  such that  $\mu_k$  vaguely converges to a random Borel measure  $\mu$  almost surely then  $\mu_k$  vaguely converges to  $\mu$  in probability.

*Remark 4.29.* It follows from Definition 4.21 that if  $\mu_k$  is a sequence of random, finite, Borel measures on  $X$  such that  $\mu_k$  weakly converges to a random, finite, Borel measure  $\mu$  in probability then  $\mu_k$  vaguely converges to  $\mu$  in probability.

**Proposition 4.30.** *The convergence vaguely in probability is induced by the topology which has base elements formed by finite intersection of sets in the form:*

$$\left\{ \mu : \rho \left( \int_X f(x) d\mu(x), Y \right) < r \right\}$$

where  $f \in C_c(X)$ ,  $r > 0$  and  $Y$  is a real-valued random variable with almost surely finite values.

**Lemma 4.31.** *Let  $X$  be locally compact. Let  $\mu$  and  $\mu_k$  be a sequence of random, locally finite, Borel measures on  $X$ . Assume that for every subsequence  $\{\alpha_k\}_{k=1}^\infty$  of  $\mathbb{N}$  there exists a subsequence  $\{\beta_k\}_{k=1}^\infty$  of  $\{\alpha_k\}_{k=1}^\infty$  and there exists an event  $H \in \mathcal{A}$  with  $P(H) = 1$  such that  $\mu_k$  vaguely converges to  $\mu$  on  $H$  then  $\int_X f(x) d\mu_k(x)$  converges to  $\int_X f(x) d\mu(x)$  in probability for every  $f \in C_c(X)$ .*

Lemma 4.31 can be proven similarly to the proof of Proposition 4.8.

**Lemma 4.32.** *Let  $X$  be locally compact. There exists  $\Psi \subseteq C_c(X)$  countable and dense subset with respect to the supremum norm such that if  $\mu_k$  is a sequence of deterministic, locally finite, Borel measures on  $X$  such that  $\int_X f(x) d\mu_k(x)$  converges to a finite limit  $S(f)$  for every  $f \in \Psi$  then  $\mu_k$  vaguely converges to a locally finite, Borel measure.*

*Proof.* Since  $X$  is a separable metric space and locally compact we can find a sequence of open sets  $G_1 \subseteq G_2 \subseteq \dots$  such that  $\bigcup_{i=1}^\infty G_i = X$  and  $\overline{G_i}$  is compact for every  $i \in \mathbb{N}$ . Hence for every compact set  $K$  there exists  $i \in \mathbb{N}$  such that  $K \subseteq G_i$ . We can further assume that  $\overline{G_i} \subseteq G_{i+1}$ . We can find  $\Psi_i \subseteq C_b(\overline{G_i})$  countable and dense subset by Lemma 3.20. Let  $\Psi = \bigcup_{i=1}^\infty \Psi_i$ .

Let  $\mu_k$  be a sequence of deterministic, locally finite, Borel measures on  $X$  such that  $\int_X f(x) d\mu_k(x)$  converges to a finite limit  $S(f)$  for every  $f \in \Psi$ . If  $\int_X f(x) d\mu_k(x)$  converges to a finite limit  $S(f)$  for every  $f \in C_c(X)$  then clearly  $S$  is a positive linear functional, hence by the Riesz-Markov theorem [4] there exists a locally finite, Borel measures  $\mu$  on  $X$  such that  $S(f) = \int_X f(x) d\mu(x)$  for every  $f \in C_c(X)$ . Hence the statement would follow. Thus to finish the proof we need to show that  $\int_X f(x) d\mu_k(x)$  converges to a finite limit  $S(f)$  for every  $f \in C_c(X)$ .

Let  $f \in C_c(X)$ , let  $K = \text{supp}(f)$ . There exists  $i \in \mathbb{N}$  such that  $K \subseteq G_i \subseteq \overline{G_i} \subseteq G_{i+1}$ . By Tietze's extension theorem we can find  $h \in C_c(X)$  such that  $h_0(x) = 1$  for  $x \in \overline{G_i}$  and  $h_0(x) = 0$  for  $x \notin G_{i+1}$ . There exists  $h \in \Psi$  such that  $\|h_0 - h\|_\infty < 1/2$ . Then  $\limsup_{k \rightarrow \infty} \mu_k(\overline{G_i}) \leq 2 \lim_{k \rightarrow \infty} \int_X h(x) d\mu_k(x) = 2S(h)$ .

Let  $g_n \in \Psi_i$  be such that  $\|f - g_n\|_\infty < 1/n$ . Then

$$\limsup_{k \rightarrow \infty} \left| \int_X f(x) d\mu_k(x) - S(g_n) \right| \leq \limsup_{k \rightarrow \infty} \left| \int_X f(x) d\mu_k(x) - \int_X g_n(x) d\mu_k(x) \right| + \limsup_{k \rightarrow \infty} \left| \int_X g_n(x) d\mu_k(x) - S(g_n) \right|$$

$$\limsup_{k \rightarrow \infty} \mu_k(\overline{G_i})/n + 0 \leq 2S(h)/n.$$

Thus for every  $n \in \mathbb{N}$  there exists  $N \in \mathbb{N}$  such that  $\left| \int_X f(x) d\mu_k(x) - S(g_n) \right| \leq 3S(h)/n$  for every  $k \geq N$  and so  $\left| \int_X f(x) d\mu_k(x) - \int_X f(x) d\mu_j(x) \right| \leq 6S(h)/n$  for every  $j, k \geq N$ . Hence  $\int_X f(x) d\mu_k(x)$  is a Cauchy sequence so it has a finite limit  $S(f)$  in  $\mathbb{R}$ .  $\square$

**Lemma 4.33.** *Let  $X$  be locally compact, let  $\Psi \subseteq C_c(X)$  as in Lemma 4.32 and  $\mu_k$  be a sequence of random, locally finite Borel measures on  $X$ . Assume that  $\int_X f(x) d\mu_k(x)$  converges in probability to a random finite limit  $S(f)$  for every  $f \in \Psi$ . Then there exists a random, locally finite Borel measures  $\mu$  on  $X$  such that for every subsequence  $\{\alpha_k\}_{k=1}^\infty$  of  $\mathbb{N}$  there exists a subsequence  $\{\beta_k\}_{k=1}^\infty$  of  $\{\alpha_k\}_{k=1}^\infty$  and there exists an event  $H \in \mathcal{A}$  with  $P(H) = 1$  such that  $\mu_k$  vaguely converges to  $\mu$  on the event  $H$ .*

Lemma 4.33 can be shown similarly to the proof of Proposition 4.10 and Theorem 4.11 by replacing the use of Lemma 4.32 by the use of Lemma 4.33.

**Theorem 4.34.** *Let  $X$  be locally compact and let  $\mu$  and  $\mu_k$  be a sequence of random, locally finite Borel measures on  $X$ . Then  $\mu_k$  vaguely converges to  $\mu$  in probability if and only if for every subsequence  $\{\alpha_k\}_{k=1}^\infty$  of  $\mathbb{N}$  there exists a subsequence  $\{\beta_k\}_{k=1}^\infty$  of  $\{\alpha_k\}_{k=1}^\infty$  and there exists an event  $H \in \mathcal{A}$  with  $P(H) = 1$  such that  $\mu_k$  vaguely converges to  $\mu$  on the event  $H$ .*

Theorem 4.34 follows from Lemma 4.31, Lemma 4.33.

**Proposition 4.35.** *Let  $X$  be locally compact. Let  $\nu$  be a deterministic Borel measure on  $X$  and  $\mu_k$  be a sequence of random, locally finite, Borel measures on  $X$  such that  $\mu_k \ll \nu$  almost surely for every  $k$ , there exists  $c > 0$  such that  $E\left(\frac{d\mu_k(x)}{d\nu(x)}\right) \leq c$  for every  $k$  and  $\mu_k(A)$  converges in probability for every Borel set  $A \subseteq X$ . Let  $f : X \rightarrow \mathbb{R}$  be a Borel measurable function such that  $\int_X |f(x)| d\nu(x) < \infty$ . Then  $\int_X f(x) d\mu_k(x)$  converges to a random variable  $Y$  in probability and  $E(|Y|) \leq c \int_X |f(x)| d\nu(x)$ .*

The proof of Proposition 4.35 is identical to the proof of Proposition 4.12.

**Proposition 4.36.** *Let  $X$  be locally compact. Let  $\mu^i$  and  $\mu_k^i$  be a sequence of random, locally finite, Borel measures on  $X$  for every  $i \in \mathbb{N}$ . Assume that  $\mu_k^i$  vaguely converges to  $\mu^i$  in probability for every  $i \in \mathbb{N}$  as  $k$  goes to  $\infty$ . Assume that for every compact set  $K \subseteq X$  and  $\varepsilon > 0$  there exist  $N, n_0 \in \mathbb{N}$  such that  $\sum_{i=N}^\infty E(\mu_k^i(K)) < \varepsilon$  for every  $k \geq n_0$ . Then there exists  $n_1 \in \mathbb{N}$  such that  $\sum_{i \in \mathbb{N}} \mu_k^i$  is a sequence of random, locally finite Borel measures, for  $k \geq n_1$ , that vaguely converges to the random, locally finite Borel measure  $\sum_{i \in \mathbb{N}} \mu^i$  in probability.*

*Proof.* Since for every compact set  $K \subseteq X$  there exists  $N_1, n_1 \in \mathbb{N}$  such that  $\sum_{i=N_1}^\infty E(\mu_k^i(K)) \leq 1$  for every  $k \geq n_1$  it follows that  $\sum_{i=N_1}^\infty \mu_k^i(K) < \infty$  almost surely for  $k \geq n_1$  and so  $\sum_{i=1}^\infty \mu_k^i(K) < \infty$  almost surely for  $k \geq n_1$ . Thus  $\sum_{i=1}^\infty \mu_k^i$  is a random, locally finite, Borel measure for  $k \geq n_1$  since  $X$  is locally compact. Let  $h \in C_c(X)$  such that  $h(x) = 1$  for every  $x \in K$  and  $\|h\|_\infty \leq 1$ , we can find such  $h$  by Tietze's extension theorem and the fact that  $X$  is locally compact. We can find  $n_2, N_2 \in \mathbb{N}$  such that  $\sum_{i=N_2}^\infty E(\mu_k^i(\text{supp}(h))) \leq 1$  for every  $k \geq n_2$ . We can find, by Lemma 3.7, a subsequence  $\{\alpha_k\}_{k=1}^\infty$  of  $\mathbb{N}$  and an event  $H \in \mathcal{A}$  with  $P(H) = 1$  such that  $\int_X h(x) d\mu_{\alpha_k}^i(x)$  converges to  $\int_X h(x) d\mu^i(x)$  on  $H$  as  $k$  goes to  $\infty$  for every  $i \in \mathbb{N}$ . Then by Fatou's lemma and Fubini's theorem

$$\begin{aligned} E\left(\sum_{i=N_2}^\infty \mu^i(K)\right) &\leq E\left(\sum_{i=N_2}^\infty \int_X h(x) d\mu^i(x)\right) = E\left(\sum_{i=N_2}^\infty \lim_{k \rightarrow \infty} \int_X h(x) d\mu_{\alpha_k}^i(x)\right) \\ &\leq \liminf_{k \rightarrow \infty} \sum_{i=N_2}^\infty E\left(\int_X h(x) d\mu_{\alpha_k}^i(x)\right) \leq \liminf_{k \rightarrow \infty} \sum_{i=N_2}^\infty E(\mu_{\alpha_k}^i(\text{supp}(h))) \leq 1. \end{aligned} \quad (18)$$

Thus  $\sum_{i=N_1}^\infty \mu^i(K) < \infty$  almost surely and so  $\sum_{i=1}^\infty \mu^i(K) < \infty$  almost surely. Hence  $\sum_{i=1}^\infty \mu^i$  is a random, locally finite, Borel measure.



Let  $f \in C_c(X)$  and  $K = \text{supp}(f)$ . Let  $\varepsilon > 0$  be fixed and let  $N, n_0 \in \mathbb{N}$  such that  $\sum_{i=N}^{\infty} E(\mu_k^i(K)) < \varepsilon$  for every  $k \geq n_0$ . Similarly to (18) we can further assume that  $E(\sum_{i=N}^{\infty} \mu^i(K)) \leq \varepsilon$ . Thus

$$\begin{aligned} & \rho\left(\sum_{i=1}^{\infty} \int_X f(x) d\mu_k^i(x), \sum_{i=1}^{\infty} \int_X f(x) d\mu^i(x)\right) \leq \\ & \rho\left(\sum_{i=1}^{\infty} \int_X f(x) d\mu_k^i(x), \sum_{i=1}^{N-1} \int_X f(x) d\mu_k^i(x)\right) + \rho\left(\sum_{i=1}^{N-1} \int_X f(x) d\mu_k^i(x), \sum_{i=1}^{N-1} \int_X f(x) d\mu^i(x)\right) + \rho\left(\sum_{i=1}^{N-1} \int_X f(x) d\mu^i(x), \sum_{i=1}^{\infty} \int_X f(x) d\mu^i(x)\right) \\ & \leq \varepsilon \|f\|_{\infty} + \rho\left(\sum_{i=1}^{N-1} \int_X f(x) d\mu_k^i(x), \sum_{i=1}^{N-1} \int_X f(x) d\mu^i(x)\right) + \varepsilon \|f\|_{\infty} \end{aligned}$$

where we used the fact that  $\rho(Y, Z) \leq E(|Y - Z|)$ . Thus

$$\limsup_{k \rightarrow \infty} \rho\left(\sum_{i=1}^{\infty} \int_X f(x) d\mu_k^i(x), \sum_{i=1}^{\infty} \int_X f(x) d\mu^i(x)\right) \leq 2\varepsilon \|f\|_{\infty}$$

since  $\sum_{i=1}^{N-1} \int_X f(x) d\mu_k^i(x)$  converges to  $\sum_{i=1}^{N-1} \int_X f(x) d\mu^i(x)$  in probability. By taking limit  $\varepsilon$  goes to 0 it follows that  $\sum_{i=1}^{\infty} \int_X f(x) d\mu_k^i(x)$  converges to  $\sum_{i=1}^{\infty} \int_X f(x) d\mu^i(x)$  in probability.  $\square$

**Theorem 4.37.** *Let  $X$  be locally compact. Let  $\nu$  be a deterministic, locally finite, Borel measure on  $X$ . Let  $\mu_k$  be a sequence of random, finite, Borel measures on  $X$  such that  $\mu_k \ll \nu$  almost surely for every  $k$ , there exists  $c > 0$  such that  $E\left(\frac{d\mu_k(x)}{d\nu(x)}\right) \leq c$  for every  $k$  and  $\mu_k(A)$  converges to a random variable  $\tau(A)$  in probability for every Borel set  $A \subseteq X$  such that  $\bar{A}$  is compact. Then  $\mu_k$  vaguely converges to a random, locally finite, Borel measure  $\mu$  in probability. Furthermore,  $\int_X f(x) d\mu_k(x)$  converges to a random variable  $S(f)$  in probability with  $E(|S(f)|) \leq c \int_X |f(x)| d\nu(x)$  for every  $f : X \rightarrow \mathbb{R}$  Borel measurable function such that  $\int_X |f(x)| d\nu(x) < \infty$ . For every countable collection of bounded, compactly supported, Borel measurable functions  $f_n : X \rightarrow \mathbb{R}$  we have that  $\int_X f_n(x) d\mu(x) = S(f_n)$  almost surely.*

*Proof.* Since  $X$  is separable metric space and locally compact and  $\nu$  is locally finite it follows that there exists a sequence of disjoint sets  $A_i$  such that  $\nu^i = \nu|_{A_i}$  is a finite measure and  $\bar{A}_i$  is compact for every  $i \in \mathbb{N}$  and  $X = \cup_{i \in \mathbb{N}} A_i$ . Let  $\mu_k^i = \mu_k|_{A_i}$ . Then  $\mu_k = \sum_{i=1}^{\infty} \mu_k^i$  and  $\mu_k^i$  vaguely converges to a random, finite Borel measure  $\mu^i$  in probability by Remark (4.29) and Theorem (4.17). Since  $\sum_{i=N}^{\infty} E(\mu_k^i(K)) \leq c \sum_{i=N}^{\infty} \nu(K) < \infty$  for every compact set  $K$  it follows that the conditions of Proposition 4.36 are satisfied and so  $\mu = \sum_{i=1}^{\infty} \mu^i$  is a random, locally finite, Borel measure and  $\mu_k$  vaguely converges to  $\mu$  in probability.

By Proposition 4.35 it follows that  $\int_X f(x) d\mu_k(x)$  converges to a random variable  $S(f)$  in probability with  $E(|S(f)|) \leq c \int_X |f(x)| d\nu(x)$  for every  $f : X \rightarrow \mathbb{R}$  Borel measurable function such that  $\int_X |f(x)| d\nu(x) < \infty$ .

Let  $\varepsilon > 0$ . For a fix  $n \in \mathbb{N}$  we can find  $g \in C_c(X)$  and an open set  $G \subseteq X$  such that  $f_n(x) = g(x)$  on  $X \setminus G$ ,  $\|g\|_\infty \leq \|f_n\|_\infty$ ,  $\nu(G) < \varepsilon$  and  $\overline{G}$  is compact by Remark 3.11. The rest of the proof proceeds similarly to the proof of Theorem 4.17.  $\square$

*Remark 4.38.* In particular, if we take  $f_n = \chi_{A_n}$  in Theorem 4.37 for a countable collection of Borel sets  $A_n$  such that  $\overline{A_n}$  is compact then it follows that  $\mu(A_n) = S(\chi_{A_n})$  almost surely for every  $n$  where  $\mu_k(A_n)$  converges to  $S(\chi_{A_n})$  in probability as  $k$  goes to infinity.

*Remark 4.39.* In Theorem 4.37 we can relax the condition  $E\left(\frac{d\mu_k(x)}{d\nu(x)}\right) \leq c$  for every  $k$ . It is enough to assume that there exists a nonnegative Borel function  $c : X \rightarrow \mathbb{R}$  such that  $\int c(x)d\nu(x) < \infty$  and  $E\left(\frac{d\mu_k(x)}{d\nu(x)}\right) \leq c(x)$  for  $\nu$  almost every  $x \in X$  for every  $k$ . It can be seen by replacing  $d\nu(x)$  by  $c(x)d\nu(x)$ .

**Proposition 4.40.** *Let  $\mu_k$  be a sequence of random, locally finite, Borel measures on  $X$  such that there exists a sequence of random closed sets  $F_1 \supseteq F_2 \supseteq \dots$  such that  $\text{supp}\mu_k \subseteq F_k$  almost surely. If  $\mu_k$  vaguely converges to a random, locally finite, Borel measure  $\mu$  in probability then  $\text{supp}\mu \subseteq \bigcap_{n=1}^\infty F_n$ .*

Proposition 4.40 can be shown similarly to the proof Proposition 4.20 due to the equivalence in Theorem 4.34.

## 5 Decomposition of measure

**Proposition 5.1.** *There exist two measures  $\nu_{\varphi R} = \nu_R$  and  $\nu_{\varphi \perp} = \nu_\perp$  with the following properties:*

- i)  $\nu = \nu_R + \nu_\perp$
- ii)  $\nu_R \perp \nu_\perp$
- iii)  $\nu_\perp$  is singular to every measure with finite  $\varphi$ -energy
- iv) there exists a sequence of sets  $(A_n)_{n \in \mathbb{N}}$  such that  $\nu_R = \nu|_{\bigcup_{n \in \mathbb{N}} A_n}$  and  $I_\varphi(\nu|_{A_n}) = I_\varphi(\nu_R|_{A_n}) < \infty$ .

*Proof.* Let  $c_{\max} = \sup\{\nu(B) : B \subseteq \bigcup B_n \text{ and } I_\alpha(\nu|_{B_n}) < \infty \text{ for all } n\}$ . We can find  $A$  and  $(A_n)_{n \in \mathbb{N}}$  such that  $\nu(A) = c_{\max}$ ,  $A = \bigcup A_n$  and  $I_\varphi(\nu|_{A_n}) < \infty$  for all  $n$ . Let  $\nu_R = \nu|_A$  and  $\nu_\perp = \nu|_{X \setminus A}$ . Then i), ii) and iv) are satisfied. Assume for a contradiction that iii) is not satisfied that is there exists a measure  $\tau$  with  $I_\varphi(\tau) < \infty$  such that  $\tau \ll \nu_\perp$ . There exists  $N > 0$  such that  $\nu_\perp(C_N) > 0$  where  $C_N = \left\{x : \frac{d\tau}{d\nu_\perp}(x) \geq \frac{1}{N}\right\}$ . If  $D \subseteq C_N$  then

$$\tau(D) = \int_D \frac{d\tau}{d\nu_\perp}(x)d\nu_\perp(x) \geq \int_D \frac{1}{N}d\nu_\perp(x) = \frac{1}{N}\nu_\perp(D).$$

Thus  $I_\varphi(\nu_\perp|_{C_N}) \leq N^2 I_\varphi(\tau|_{C_N}) \leq N^2 I_\varphi(\tau) < \infty$ . This contradicts with the maximality of  $c_{\max}$ .  $\square$

*Notation 5.2.* We call  $\nu_R$  the  $\varphi$ -regular part of  $\nu$  and we call  $\nu_\perp$  the  $\varphi$ -singular part of  $\nu$ . These are uniquely determined by  $\nu$  and  $\varphi$ .

## 6 Degenerate case

**Proposition 6.1.** *Let  $\tau$  be a finite Borel measure on  $X$  such that  $\int \int \varphi(x, y) d\tau(y) d\tau(x) < \infty$ . Then for every  $\varepsilon > 0$  there exist  $0 < M < \infty$  and a compact set  $F \subseteq X$  such that  $\tau(X \setminus F) < \varepsilon$  and  $\int_F \varphi(x, y) d\tau(y) < M$  for every  $x \in X$ .*

*Proof.* Since  $\int \int \varphi(x, y) d\tau(y) d\tau(x) < \infty$  it follows that  $\int \varphi(x, y) d\tau(y) < \infty$  for  $\tau$  almost every  $x \in X$ . Thus there exists a compact set  $F_0 \subseteq X$  and  $M_0 > 0$  such that  $\tau(X \setminus F_0) < \varepsilon/2$  and  $\int \varphi(x, y) d\tau(y) < M_0$  for every  $x \in F_0$ . In the proof of [5, Chapter III. Thm 1, page 15] it is shown that there exists a compact set  $F \subseteq F_0$  such that  $\tau(X \setminus F) < \varepsilon$  and  $\lim_{x \rightarrow x_0} \int_F \varphi(x, y) d\tau(y) = \int_F \varphi(x_0, y) d\tau(y) < M_0$  for every  $x_0 \in F$ . Hence there exists  $r > 0$  such that if  $\text{dist}(F, x) < r$  then  $\int_F \varphi(x, y) d\tau(y) < M_0$ . Whenever  $\text{dist}(F, x) \geq r$  then  $\int_F \varphi(x, y) d\tau(y) \leq \varphi(r) \tau(X)$ . Hence the statement follows.  $\square$

**Proposition 6.2.** *If  $\nu$  is a finite Borel measure and  $I_\varphi(\nu|_A) = \infty$  for every measurable set  $A$  of positive  $\nu$ -measure then there exists a Borel set  $Z \subseteq X$  such that  $\nu(X \setminus Z) = 0$  and  $C_\varphi(Z) = 0$ .*

*Proof.* Let  $A_n = \{x \in X : \int \varphi(x, y) d\nu(y) \leq n\}$ . Then

$$I_\varphi(\nu|_{A_n}) = \int_{A_n} \int_{A_n} \varphi(x, y) d\nu(y) d\nu(x) \leq \int_{A_n} \int_X \varphi(x, y) d\nu(y) d\nu(x) \leq n \cdot \nu(A_n) < \infty.$$

Thus by the assumption  $\nu(A_n) = 0$  and hence  $\nu(X \setminus Z) = 0$  for  $Z = \{x \in X : \int \varphi(x, y) d\nu(y) = \infty\}$ .

Assume for a contradiction that  $C_\varphi(Z) > 0$ . Then there exists a probability measure  $\tau$  on  $Z$  such that  $\int \int \varphi(x, y) d\tau(y) d\tau(x) < \infty$ . Then by Lemma 6.1 there exist  $0 < M < \infty$  and  $F \subseteq Z$  such that  $\tau(F) > 0$  and  $\int_F \varphi(x, y) d\tau(y) < M$  for every  $x \in X$ . Thus

$$\int_X \left( \int_F \varphi(x, y) d\tau(y) \right) d\nu(x) \leq \int_X M d\nu(x) \leq M \cdot \nu(X) < \infty$$

contradicting with that

$$\int_X \left( \int_F \varphi(x, y) d\tau(y) \right) d\nu(x) = \int_F \left( \int_X \varphi(x, y) d\nu(x) \right) d\tau(y) = \int_F \infty d\tau(x) = \infty.$$

Hence  $C_\varphi(Z) = 0$ .  $\square$

**Lemma 6.3.** *Let  $F \subseteq X$  be a compact set and let  $B_\omega$  be a closed realisation of the random set  $B$  such that  $F \cap B_\omega = \emptyset$ . Then  $\mathcal{C}_k(\nu)(X) = \mu_k(X) = \mu_{k,\omega}(X)$  converges to 0 for that realisation  $B_\omega$ .*

*Proof.* Since  $B \cap F = \emptyset$  then  $\text{dist}(B, F) > 0$ . Let  $k_0$  be such that  $\sup \{\text{diam}(Q) : Q \in \mathcal{Q}_k\} < \text{dist}(B, F)$  for every  $k \geq k_0$  (we note that  $k_0$  depends on the realisation  $B_\omega$  but exists nevertheless). Then  $\mu_{k,\omega}(X) = \mathcal{C}_{k,\omega}(\nu)(X) = 0$  by the definition of  $\mathcal{C}_k$  for  $k \geq k_0$ .  $\square$

**Theorem 6.4.** *Assume that  $B$  is almost surely a closed set and if  $C_\varphi(F) = 0$  for some compact set  $F \subseteq X$  then  $B \cap F = \emptyset$  almost surely. If  $\nu$  is a finite Borel measure and  $I_\varphi(\nu|_A) = \infty$  for every Borel set  $A$  of positive  $\nu$ -measure then  $\mathcal{C}_k(\nu)(X) = \mu_k(X)$  converges to 0 in probability.*

*Proof.* Let  $\varepsilon > 0$  be fixed. By Proposition 6.2 there exists  $Z \subseteq X$  such that  $\nu(X \setminus Z) = 0$  and  $C_\varphi(Z) = 0$ . Let  $\eta > 0$  be fixed. Since  $\nu$  is a finite Borel measure (reference for inner regularity) there exists a compact set  $F \subseteq Z$  such that  $\nu(X \setminus F) < \eta$ . Then by assumption  $B \cap F = \emptyset$  almost surely. Thus  $\mathcal{C}_k(\nu|_F)(X)$  converges to 0 almost surely by Lemma 6.3 and hence converges to 0 in probability. Since  $\mu_k = \mathcal{C}_k(\nu) = \mathcal{C}_k(\nu|_F) + \mathcal{C}_k(\nu|_{X \setminus F})$  almost surely it follows that

$$\begin{aligned} \limsup_{k \rightarrow \infty} \rho(\mathcal{C}_k(\nu)(X), 0) &\leq \limsup_{k \rightarrow \infty} \rho(\mathcal{C}_k(\nu)(X), \mathcal{C}_k(\nu|_F)(X)) + \rho(\mathcal{C}_k(\nu|_F)(X), 0) \\ &\leq \limsup_{k \rightarrow \infty} E(\mathcal{C}_k(\nu|_{X \setminus F})(X)) + 0 \leq \nu(X \setminus F) < \eta. \end{aligned}$$

Since we can choose  $\eta$  to be arbitrarily small it follows that  $\lim_{k \rightarrow \infty} \rho(\mathcal{C}_k(\nu)(X), 0) = 0$ .  $\square$

## 7 $\mathcal{L}^2$ -boundedness

**Lemma 7.1.** *Let  $Q, S \in \mathcal{Q}_k$ ,  $Q \neq S$  and  $\nu_1$  and  $\nu_2$  be continuous, finite Borel measures. If (16) holds for  $Q$  and  $S$  then*

$$E(\mathcal{C}_k(\nu_1)(Q) \cdot \mathcal{C}_k(\nu_2)(S)) \leq c \cdot \varphi(\text{dist}(Q, S)) \cdot \nu_1(Q) \cdot \nu_2(S).$$

*Proof.* By the definition of  $\mathcal{C}_k(\nu)$  and (16) it follows that

$$\begin{aligned} &E(\mathcal{C}_k(\nu_1)(Q) \cdot \mathcal{C}_k(\nu_2)(S)) \\ &= E(P(Q \cap B \neq \emptyset)^{-1} \cdot I_{Q \cap B \neq \emptyset} \cdot \nu_1(Q) \cdot P(S \cap B \neq \emptyset)^{-1} \cdot I_{S \cap B \neq \emptyset} \cdot \nu_2(S)) \\ &= P(Q \cap B \neq \emptyset)^{-1} \cdot P(S \cap B \neq \emptyset)^{-1} \cdot P(Q \cap B \neq \emptyset \text{ and } S \cap B \neq \emptyset) \cdot \nu_1(Q) \cdot \nu_2(S) \\ &\leq c \cdot \varphi(\text{dist}(Q, S)) \cdot \nu_1(Q) \cdot \nu_2(S). \end{aligned}$$

$\square$

**Lemma 7.2.** *Let  $\nu_1$  and  $\nu_2$  be continuous, finite Borel measures. If  $Q, S \in \mathcal{Q}_k$  such that  $\max\{\text{diam}(Q), \text{diam}(S)\} < \delta \cdot \text{dist}(Q, S)$  then*

$$\varphi(\text{dist}(Q, S)) \cdot \nu_1(Q) \cdot \nu_2(S) \leq c_2 \int_S \left( \int_Q \varphi(x, y) d\nu_1(x) \right) d\nu_2(y) + c_3 \nu_1(Q) \nu_2(S).$$

*Proof.* If  $x \in Q$  and  $y \in S$  then

$$d(x, y) \leq \text{dist}(Q, S) + \text{diam}(Q) + \text{diam}(S) < \text{dist}(Q, S) \cdot (1 + 2\delta).$$

Then by (8) and the monotonicity of  $\varphi$  it follows that

$$\varphi(\text{dist}(Q, S)) \leq c_2 \varphi(\text{dist}(Q, S) \cdot (1 + 2\delta)) + c_3 \leq c_2 \varphi(d(x, y)) + c_3.$$

Integrating on  $Q \times S$  with respect to  $\nu_1 \times \nu_2$  the statement follows.  $\square$

**Lemma 7.3.** Assume that (15) holds. Let  $\nu_1$  and  $\nu_2$  be finite Borel measures. If  $Q, S \in \mathcal{Q}_k$  then

$$E(\mathcal{C}_k(\nu_1)(Q) \cdot \mathcal{C}_k(\nu_2)(S)) \leq a^{-1} \left( \int_Q \int_Q \varphi(x, y) d\nu_1(x) d\nu_1(y) + \int_S \int_S \varphi(x, y) d\nu_2(x) d\nu_2(y) \right).$$

*Proof.* Due to symmetry without the loss of generality we can assume that  $\nu_1(Q) \leq \nu_2(S)$ . By the definition of  $\mathcal{C}_k$  and (15) it follows that

$$\begin{aligned} & E(\mathcal{C}_k(\nu_1)(Q) \cdot \mathcal{C}_k(\nu_2)(S)) \\ &= E(P(Q \cap B \neq \emptyset)^{-1} \cdot I_{Q \cap B \neq \emptyset} \cdot \nu_1(Q) \cdot P(S \cap B \neq \emptyset)^{-1} \cdot I_{S \cap B \neq \emptyset} \cdot \nu_2(S)) \\ &= P(Q \cap B \neq \emptyset)^{-1} \cdot P(S \cap B \neq \emptyset)^{-1} \cdot P(Q \cap B \neq \emptyset \text{ and } S \cap B \neq \emptyset) \cdot \nu_1(Q) \cdot \nu_2(S) \\ &\leq P(Q \cap B \neq \emptyset)^{-1} \cdot P(S \cap B \neq \emptyset)^{-1} \cdot P(Q \cap B \neq \emptyset) \cdot \nu_2(S) \cdot \nu_2(S) \\ &= P(S \cap B \neq \emptyset)^{-1} \cdot \nu_2(S)^2 \leq a^{-1} \cdot C_\varphi(S)^{-1} \cdot \nu_2(S)^2 \leq a^{-1} \cdot (I_\varphi(\nu|_S) \cdot \nu(S)^{-2}) \cdot \nu_2(S)^2 \\ &= a^{-1} \int_S \int_S \varphi(x, y) d\nu_2(x) d\nu_2(y) \leq a^{-1} \left( \int_Q \int_Q \varphi(x, y) d\nu_1(x) d\nu_1(y) + \int_S \int_S \varphi(x, y) d\nu_2(x) d\nu_2(y) \right). \end{aligned}$$

□

**Proposition 7.4.** Assume that (15) holds and there exists  $\delta > 0$  such that (16) and (13). If  $\nu$  is a finite Borel measure on  $X$  and  $I_\varphi(\nu) < \infty$  then

$$E(\mathcal{C}_k(\nu)(X) \mathcal{C}_k(\nu)(X)) \leq (cc_2 + 2a^{-1}M_\delta) I_\varphi(\nu) + cc_3\nu(X)^2$$

for every  $k \in \mathbb{N}$ .

*Proof.* We say that a pair  $(Q, S)$  is a ‘good’ pair if  $\max\{\text{diam}(Q), \text{diam}(S)\} < \delta \cdot \text{dist}(Q, S)$  and is a bad pair if  $\max\{\text{diam}(Q), \text{diam}(S)\} \geq \delta \cdot \text{dist}(Q, S)$ . Combining Lemma (7.1) and Lemma (7.2) it follows that

$$\begin{aligned} \sum_{(Q, S) \text{ is good}} E(\mathcal{C}_k(\nu)(X) \mathcal{C}_k(\nu)(X)) &\leq \sum_{(Q, S) \text{ is good}} c \cdot \varphi(\text{dist}(Q, S) \cdot \nu(Q) \cdot \nu(S)) \\ &\leq c \sum_{(Q, S) \text{ is good}} \left( c_2 \int_S \left( \int_Q \varphi(x, y) d\nu(x) \right) d\nu(y) + c_3 \nu(Q) \nu(S) \right) \\ &\leq c \sum_{Q, S \in \mathcal{Q}_k} \left( c_2 \int_S \left( \int_Q \varphi(x, y) d\nu(x) \right) d\nu(y) + c_3 \nu(Q) \nu(S) \right) \leq c (c_2 I_\varphi(\nu) + c_3 \nu(X)^2). \end{aligned}$$

By Lemma (7.3) and (13)

$$\begin{aligned}
& \sum_{(Q,S) \text{ is bad}} E(\mathcal{C}_k(\nu)(Q)\mathcal{C}_k(\nu)(S)) \\
& \leq \sum_{(Q,S) \text{ is bad}} a^{-1} \left( \int_Q \int_Q \varphi(x,y) d\nu(x) d\nu(y) + \int_S \int_S \varphi(x,y) d\nu(x) d\nu(y) \right) \\
& = 2a^{-1} \sum_{Q \in \mathcal{Q}_k} \left( \sum_{\substack{S \in \mathcal{Q}_k \\ (Q,S) \text{ is bad}}} \int_Q \int_Q \varphi(x,y) d\nu(x) d\nu(y) \right) \\
& \leq 2a^{-1} \sum_{Q \in \mathcal{Q}_k} M_\delta \int_Q \int_Q \varphi(x,y) d\nu(x) d\nu(y) \leq 2a^{-1} M_\delta I_\varphi(\nu).
\end{aligned}$$

Hence the statement follows by

$$\begin{aligned}
E(\mu_k(X)^2) &= \sum_{Q,S \in \mathcal{Q}_k} E(\mu_k(Q)\mu_k(S)) \\
&= \sum_{(Q,S) \text{ is good}} E(\mu_k(Q)\mu_k(S)) + \sum_{(Q,S) \text{ is bad}} E(\mu_k(Q)\mu_k(S)) \\
&\leq c(c_2 I_\varphi(\nu) + c_3 \nu(X)^2) + 2a^{-1} M_\delta I_\varphi(\nu).
\end{aligned}$$

□

## 8 Non-degenerate limit

*Notation 8.1.* For  $x \in X$  and  $k \in \mathbb{N}$  let  $Q_k(x) = Q$  if  $x \in Q$  for some  $Q \in \mathcal{Q}_k$  and  $Q_k(x) = \emptyset$  otherwise. There is at most one such  $Q$  since elements of  $\mathcal{Q}_k$  are disjoint hence  $Q_k(x)$  is well-defined.

**Definition 8.2.** For  $k, n \in \mathbb{N}$  let  $F_{k,n} : X \times X \rightarrow \mathbb{R}$  be the nonnegative function

$$F_{k,n}(x, y) = \begin{cases} \frac{P(Q_k(x) \cap B \neq \emptyset \text{ and } Q_n(y) \cap B \neq \emptyset)}{P(Q_k(x) \cap B \neq \emptyset) \cdot P(Q_n(y) \cap B \neq \emptyset)} & \text{if } Q_k(x) \neq \emptyset, Q_n(y) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}.$$

**Lemma 8.3.** Let  $\nu_1$  and  $\nu_2$  be continuous finite Borel measures. For  $k, n \in \mathbb{N}$

$$E(\mathcal{C}_k(\nu_1)(X) \cdot \mathcal{C}_n(\nu_2)(X)) = \int_X \int_X F_{k,n}(x, y) d\nu_1(x) d\nu_2(y).$$

*Proof.* By the definition of  $\mathcal{C}_k$  it follows that

$$\begin{aligned} E(\mathcal{C}_k(\nu_1)(X) \cdot \mathcal{C}_n(\nu_2)(X)) &= \sum_{Q \in \mathcal{Q}_k} \sum_{F \in \mathcal{Q}_n} \frac{P(Q \cap B \neq \emptyset \text{ and } F \cap B \neq \emptyset)}{P(Q \cap B \neq \emptyset) \cdot P(F \cap B \neq \emptyset)} \cdot \nu_1(Q) \cdot \nu_2(F) \\ &= \sum_{Q \in \mathcal{Q}_k} \sum_{F \in \mathcal{Q}_n} \int_Q \int_F F_{k,n}(x, y) d\nu_1(x) d\nu_2(y) = \int_X \int_X F_{k,n}(x, y) d\nu_1(x) d\nu_2(y). \end{aligned}$$

□

**Lemma 8.4.** *Let  $\nu_1$  and  $\nu_2$  be continuous finite Borel measures. Let  $Q, F \subseteq X$  be Borel sets. For  $k, n \in \mathbb{N}$ , it follows that*

$$E(\mathcal{C}_k(\nu_1)(Q) \cdot \mathcal{C}_n(\nu_2)(F)) = \int_F \left( \int_Q F_{k,n}(x, y) d\nu_1(x) \right) d\nu_2(y).$$

Lemma (8.4) follows by the application of Lemma (8.3) to the measures  $\nu_1|_Q$  and  $\nu_2|_F$ .

**Definition 8.5.** We define the following functions

$$\overline{F}_N(x, y) = \sup_{n, k \geq N} F_{k,n}(x, y)$$

and

$$\underline{F}_N(x, y) = \inf_{n, k \geq N} F_{k,n}(x, y)$$

and their limits

$$\overline{F}(x, y) = \limsup_{N \rightarrow \infty} \overline{F}_N(x, y)$$

and

$$\underline{F}(x, y) = \liminf_{N \rightarrow \infty} \underline{F}_N(x, y).$$

*Remark 8.6.* If (16) holds for  $\delta > 0$  then

$$\underline{F}(x, y) \leq \overline{F}(x, y) \leq c$$

for  $x \neq y$ .

**Lemma 8.7.** *Let  $\nu_1$  and  $\nu_2$  be continuous finite Borel measures. Assume that (15) holds for every  $Q \in \mathcal{Q}_k$ ,  $k \geq 1$  and that there exists  $\delta > 0$  such that (16). Let  $\varepsilon, \eta > 0$  and let  $A_\varepsilon = \{(x, y) \in X \times X : d(x, y) > \varepsilon\}$ . Then there exists  $m \in \mathbb{N}$  such that for every  $n, k \geq m$*

$$-\eta + \iint_{A_\varepsilon} \underline{F}(x, y) d\nu_1(x) d\nu_2(y) \leq \iint_{A_\varepsilon} F_{k,n}(x, y) d\nu_1(x) d\nu_2(y) \leq \eta + \iint_{A_\varepsilon} \overline{F}(x, y) d\nu_1(x) d\nu_2(y)$$

*Proof.* Let  $k_0 \in \mathbb{N}$  be large enough that  $\sup \{diam(Q) : Q \in \mathcal{Q}_k\} < \varepsilon/3$  for every  $k > k_0$ , we can choose such  $k_0$  due to (9). Whenever  $A_\varepsilon \cap Q \times F \neq \emptyset$  for  $Q \in \mathcal{Q}_k$ ,  $F \in \mathcal{Q}_n$ ,  $k, n \geq k_0$  then  $dist(Q, F) > \varepsilon/3$ . We can choose  $k_1 \geq k_0$  such that  $\sup \{diam(Q) : Q \in \mathcal{Q}_k\} < \delta\varepsilon/3$  for every  $k \geq k_1$ . Whenever  $A_\varepsilon \cap Q \times F \neq \emptyset$  for  $Q \in \mathcal{Q}_k$ ,  $F \in \mathcal{Q}_n$ ,  $k, n \geq k_1$  then  $\max(diam(Q), diam(F)) < \delta\varepsilon/3 < \delta dist(Q, F)$ . Hence  $F_{k,n}(x, y) \leq c\varphi(dist(Q, F))$  for every  $(x, y) \in Q \times F$  by (16). Since  $\varphi$  is a nonnegative, monotone decreasing, continuous function it follows that  $\varphi$  is absolutely continuous on  $[\varepsilon/3, \infty)$ , hence we can choose  $k_2 \geq k_1$ , due to (9), such that  $\varphi(dist(Q, S)) \leq \varphi(d(x, y)) + \eta/c$  whenever  $A_\varepsilon \cap Q \times F \neq \emptyset$  for  $x \in Q \in \mathcal{Q}_k$ ,  $y \in F \in \mathcal{Q}_n$ ,  $k, n \geq k_2$ . Hence

$$F_{k,n}(x, y) \leq c\varphi(d(x, y)) + \eta$$

whenever  $A_\varepsilon \cap Q \times F \neq \emptyset$  for  $x \in Q \in \mathcal{Q}_k$ ,  $y \in F \in \mathcal{Q}_n$ ,  $k, n \geq k_2$ . Thus

$$\underline{F}_N(x, y) \leq F_{k,n}(x, y) \leq \overline{F}_N \leq c \cdot \varphi(d(x, y)) + \eta \leq c\varphi(\varepsilon) + \eta$$

for  $(x, y) \in A_\varepsilon$ ,  $N \geq k_2$ . Since  $\underline{F}_N$  and  $\overline{F}_N$  converge in  $N$ , due to the dominated convergence theorem, there exists  $m \geq k_2$  such that

$$-\eta + \iint_{A_\varepsilon} \underline{F}(x, y) d\nu_1(x) d\nu_2(y) \leq \iint_{A_\varepsilon} \underline{F}_N(x, y) d\nu_1(x) d\nu_2(y)$$

and

$$\iint_{A_\varepsilon} \overline{F}_N(x, y) d\nu_1(x) d\nu_2(y) \leq \eta + \iint_{A_\varepsilon} \overline{F}(x, y) d\nu_1(x) d\nu_2(y)$$

for every  $N \geq m$ . Thus the statement follows.  $\square$

*Notation 8.8.* For  $\varepsilon > 0$  let  $k_\varepsilon$  be the largest integer such that  $\inf \{diam(Q) : Q \in \mathcal{Q}_{k_\varepsilon}\} > \varepsilon$ , it is well-defined by (14). Let  $r_\varepsilon = \sup \{diam(Q) : Q \in \mathcal{Q}_{k_\varepsilon}\}$ . Note that  $r_\varepsilon$  converges to 0 as  $\varepsilon$  approaches 0 by (9) and (14).

**Lemma 8.9.** *Let  $\nu_1$  and  $\nu_2$  be continuous finite Borel measures and  $\nu = \nu_1 + \nu_2$ . Assume that  $I_\varphi(\nu) < \infty$ . Let  $G_\varepsilon = \{(x, y) \in X \times X : d(x, y) \leq \varepsilon\}$  and  $H_\varepsilon = \{(x, y) \in X \times X : d(x, y) \leq \varepsilon + 2r_\varepsilon\}$ . Assume that (15) holds and there exists  $0 < \delta < 1$  such that (16) and (13). Then there exists  $c_4 > 0$ , depending on  $a, c, c_2, c_3$  and  $M_\delta$ , such that*

$$\iint_{G_\varepsilon} F_{n,n}(x, y) d\nu_1(x) d\nu_2(y) \leq c_4 \iint_{H_\varepsilon} \varphi(x, y) d\nu(x) d\nu(y) + c_4 \cdot \nu \times \nu(H_\varepsilon).$$

*Proof.* Whenever  $Q \times S \cap G_\varepsilon \neq \emptyset$  for  $Q, S \in \mathcal{Q}_{k_\varepsilon}$  then  $dist(Q, S) \leq \varepsilon < \max \{diam(Q), diam(F)\}$ . In particular,  $\delta \cdot dist(Q, S) < \max \{diam(Q), diam(F)\}$ . Hence

$$\# \{S \in \mathcal{Q}_{k_\varepsilon} : Q \times S \cap G_\varepsilon \neq \emptyset\} \leq M_\delta \tag{19}$$

for every  $Q \in \mathcal{Q}_{k_\varepsilon}$  by (13). By Lemma (8.4) and Lemma (7.4) it follows that

$$\iint_{G_\varepsilon} F_{n,n}(x, y) d\nu_1(x) d\nu_2(y) \leq \iint_{G_\varepsilon} F_{n,n}(x, y) d\nu(x) d\nu(y)$$



$$\begin{aligned}
&\leq \sum_{\substack{Q, S \in \mathcal{Q}_{k_\varepsilon} \\ Q \times S \cap G_\varepsilon \neq \emptyset}} \iint_{Q \times S} F_{n,n}(x, y) d\nu(x) d\nu(y) = \sum_{\substack{Q, S \in \mathcal{Q}_{k_\varepsilon} \\ Q \times S \cap G_\varepsilon \neq \emptyset}} E(\mathcal{C}_n(\nu)(Q) \cdot \mathcal{C}_n(\nu)(S)) \\
&\leq \sum_{\substack{Q, S \in \mathcal{Q}_{k_\varepsilon} \\ Q \times S \cap G_\varepsilon \neq \emptyset}} (cc_2 + 2a^{-1}M_\delta) I_\varphi(\nu|_Q + \nu|_S) + cc_3\nu(Q \cup S)^2. \tag{20}
\end{aligned}$$

Whenever  $Q \times S \cap G_\varepsilon \neq \emptyset$  for  $Q, S \in \mathcal{Q}_{k_\varepsilon}$  then  $Q \times S \subseteq H_\varepsilon$ . Hence

$$\sum_{\substack{Q, S \in \mathcal{Q}_{k_\varepsilon} \\ Q \times S \cap G_\varepsilon \neq \emptyset}} \iint_{Q \times S} \varphi(x, y) d\nu(x) d\nu(y) \leq \iint_{H_\varepsilon} \varphi(x, y) d\nu(x) d\nu(y). \tag{21}$$

By (19)

$$\sum_{\substack{Q, S \in \mathcal{Q}_{k_\varepsilon} \\ Q \times S \cap G_\varepsilon \neq \emptyset}} \iint_{Q \times Q} \varphi(x, y) d\nu(x) d\nu(y) \leq M_\delta \sum_{Q \in \mathcal{Q}_{k_\varepsilon}} \iint_{Q \times Q} \varphi(x, y) d\nu(x) d\nu(y) \leq M_\delta \iint_{H_\varepsilon} \varphi(x, y) d\nu(x) d\nu(y) \tag{22}$$

and similarly

$$\sum_{\substack{Q, S \in \mathcal{Q}_{k_\varepsilon} \\ Q \times S \cap G_\varepsilon \neq \emptyset}} \nu(Q)^2 \leq M_\delta \sum_{Q \in \mathcal{Q}_{k_\varepsilon}} \nu(Q)^2 \leq M_\delta \cdot \nu \times \nu(H_\varepsilon) \tag{23}$$

Finally,

$$\sum_{\substack{Q, S \in \mathcal{Q}_{k_\varepsilon} \\ Q \times S \cap G_\varepsilon \neq \emptyset}} \nu(Q)\nu(S) \leq \nu \times \nu(H_\varepsilon). \tag{24}$$

Since

$$I_\varphi(\nu|_Q + \nu|_S) = 2 \iint_{Q \times S} \varphi(x, y) d\nu(x) d\nu(y) + \iint_{Q \times Q} \varphi(x, y) d\nu(x) d\nu(y) + \iint_{S \times S} \varphi(x, y) d\nu(x) d\nu(y)$$

and

$$\nu(Q \cup S)^2 = \nu(Q)^2 + \nu(S)^2 + 2\nu(Q)\nu(S),$$

the statement follows by combining (20), (21), (22), (23) and (24).  $\square$

## 8.1 Limit in the presence of the weighted kernel

**Lemma 8.10.** *Let  $\nu$  be continuous finite Borel measure. Then*

$$\int_X \int_X F_{n,n}(x, y) + F_{k,k}(x, y) - 2F_{k,n}(x, y) d\nu(x) d\nu(y) \geq 0 \quad (25)$$

for every  $k, n \in \mathbb{N}$ . If

$$\limsup_{n \rightarrow \infty} \limsup_{k \rightarrow \infty} \int_X \int_X F_{n,n}(x, y) + F_{k,k}(x, y) - 2F_{k,n}(x, y) d\nu(x) d\nu(y) = 0$$

then  $\mu_k(X) = \mathcal{C}_k(\nu)(X)$  converges in  $\mathcal{L}^2$ .

*Proof.* Since  $\mathcal{L}^2$  is complete it is sufficient to show that  $\mu_k(X)$  is a Cauchy sequence in  $\mathcal{L}^2$ . For the integers  $n \leq k$  by Lemma (8.3)

$$\begin{aligned} E((\mu_k(X) - \mu_n(X))^2) &= E(\mu_k(X)^2 + \mu_n(X)^2 - 2\mu_k(X)\mu_n(X)) \\ &= \int_X \int_X F_{k,k}(x, y) + F_{n,n}(x, y) d\nu(x) d\nu(y) - 2 \int_X \int_X F_{k,n}(x, y) d\nu(x) d\nu(y), \end{aligned}$$

hence (25) holds and by the assumption

$$\limsup_{n \rightarrow \infty} \limsup_{k \rightarrow \infty} E(\mu_k(X) - \mu_n(X))^2 = 0$$

and so  $\mu_k(X)$  is a Cauchy sequence in  $\mathcal{L}^2$ . □

**Proposition 8.11.** *Let  $\nu$  be continuous finite Borel measure with  $I_\varphi(\nu) < \infty$ . Assume that (15) holds for every  $Q \in \mathcal{Q}_k$ ,  $k \geq 1$  and that there exists  $0 < \delta < 1$  such that (16). If  $\underline{F}(x, y) = \overline{F}(x, y)$  for  $\nu \times \nu$  almsot every  $(x, y)$  then*

$$\limsup_{n \rightarrow \infty} \limsup_{k \rightarrow \infty} \int_X \int_X F_{n,n}(x, y) + F_{k,k}(x, y) - 2F_{k,n}(x, y) d\nu(x) d\nu(y) = 0.$$

*Proof.* Let  $\eta, \varepsilon > 0$ . Let  $m \in \mathbb{N}$  as in Lemma (8.7) and  $k, n \geq m$ . Let  $A_\varepsilon$ ,  $G_\varepsilon$  and  $H_\varepsilon$  as in Lemma (8.7) and Lemma (8.9). Then

$$\begin{aligned} & \int_X \int_X F_{n,n}(x, y) + F_{k,k}(x, y) - 2F_{k,n}(x, y) d\nu(x) d\nu(y) \\ & \leq \int_X \int_X F_{n,n}(x, y) + F_{k,k}(x, y) d\nu(x) d\nu(y) - 2 \iint_{A_\varepsilon} F_{k,n}(x, y) d\nu(x) d\nu(y) \\ & \leq 2\eta + 2 \iint_{A_\varepsilon} \overline{F}(x, y) d\nu(x) d\nu(y) + \iint_{G_\varepsilon} F_{n,n}(x, y) + F_{k,k}(x, y) d\nu(x) d\nu(y) + 2\eta - 2 \iint_{A_\varepsilon} \underline{F}(x, y) d\nu(x) d\nu(y) \end{aligned}$$

$$\leq 4\eta + 2 \cdot c_4 \cdot 4 \left( \iint_{H_\varepsilon} \varphi(x, y) d\nu(x) d\nu(y) + \nu \times \nu(H_\varepsilon) \right)$$

where we used Lemma (8.9) for  $\nu_1 = \nu_2 = \nu$ . Thus

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \limsup_{k \rightarrow \infty} \int_X \int_X F_{n,n}(x, y) + F_{k,k}(x, y) - 2F_{k,n}(x, y) d\nu(x) d\nu(y) \\ & \leq 4\eta + 2 \cdot c_4 \cdot 4 \left( \iint_{H_\varepsilon} \varphi(x, y) d\nu(x) d\nu(y) + \nu \times \nu(H_\varepsilon) \right). \end{aligned} \quad (26)$$

By Fubini's theorem  $\nu \times \nu((x, x) : x \in X) = 0$ . Since  $\nu \times \nu$  and  $\varphi(x, y) d\nu(x) d\nu(y)$  are finite measures it follows that

$$\iint_{H_\varepsilon} \varphi(x, y) d\nu(x) d\nu(y) + \nu \times \nu(H_\varepsilon)$$

converges to 0 as  $\varepsilon$  approaches 0. Taking the limit  $\eta$  and  $\varepsilon$  go to 0 in (26) the statement follows by (25).  $\square$

**Theorem 8.12.** *Let  $\nu$  be continuous finite Borel measure with  $I_\varphi(\nu) < \infty$ . Assume that (15) holds for every  $Q \in \mathcal{Q}_k$ ,  $k \geq 1$  and that there exists  $0 < \delta < 1$  such that (16). If  $\underline{F}(x, y) = \overline{F}(x, y)$  for  $\nu \times \nu$  almsot every  $(x, y)$  and  $A \subseteq X$  is a Borel set then  $\mu_k(A) = \mathcal{C}_k(\nu)(A)$  converges to a limit  $\mu(A)$  in  $\mathcal{L}^2$ , in  $\mathcal{L}^1$  and in probability. It follows that  $E(\mu(A)) = E(\mu_k(A)) = \nu(A)$ .*

Proposition 8.12 follows by applying Lemma 8.10 and Proposition 8.11 to the measure  $\nu|_A$ .

## 8.2 Limit in the presence of a martingale filtration

**Theorem 8.13.** *Let  $\nu$  be continuous finite Borel measure. Assume that (15) holds for every  $Q \in \mathcal{Q}_k$ ,  $k \geq 1$  and that there exists  $0 < \delta < 1$  such that (16). Assume that for every  $A \subseteq X$  there exists a filtration  $\mathcal{F}_k$  such that  $\mathcal{C}_k(\nu)(A)$  is a martingale with respect to the filtration  $\mathcal{F}_k$ . Then  $\mu_k(A)$  converges to a limit  $\mu(A)$  almost surely and  $E(\mu(A)) \leq E(\mu_k(A)) = \nu(A)$ . If additionally  $I_\varphi(\nu) < \infty$  then  $\mu_k(A)$  converges in  $\mathcal{L}^2$  and in  $\mathcal{L}^1$ , and  $E(\mu(A)) = E(\mu_k(A)) = \nu(A)$ .*

*Proof.* Since  $\mu_k(A)$  is a nonnegative martingale it converges almost surely to a random limit  $\nu(A)$  and  $E(\mu(A)) \leq E(\mu_k(A)) = \nu(A)$ . If  $I_\varphi(\nu) < \infty$  then  $\mu_k(A)$  is  $\mathcal{L}^2$ -bounded by 7.4 hence converges in  $\mathcal{L}^2$  and in  $\mathcal{L}^1$ .  $\square$

## 9 Existence of the conditional measure

**Proposition 9.1.** *Let  $\nu = \sum_{i=1}^{\infty} \nu^i$  be a finite Borel measure such that  $\mathcal{C}_k(\nu^i)(A) = \mu_k^i(A)$  converges in  $\mathcal{L}^1$  to a random limit  $\mu^i(A)$  for every  $i \in \mathbb{N}$ . Then  $\mathcal{C}_k(\nu)(A) = \sum_{i=1}^{\infty} \mu_k^i(A)$  converges in  $\mathcal{L}^1$  to  $\sum_{i=1}^{\infty} \mu^i(A)$ .*

*Proof.* Let  $\eta > 0$  and let  $n \in \mathbb{N}$  large enough that  $\sum_{i=n}^{\infty} \nu^i(A) < \eta$ . By assumption  $\sum_{i=1}^n \mu_k^i(A)$  converges in  $\mathcal{L}^1$  to  $\sum_{i=1}^n \mu^i(A)$ , hence converges in probability. Thus

$$\begin{aligned} \limsup_{k \rightarrow \infty} P \left( \left| \sum_{i=1}^{\infty} \mu_k^i(A) - \sum_{i=1}^{\infty} \mu^i(A) \right| > 2\varepsilon \right) &\leq \limsup_{k \rightarrow \infty} P \left( \left| \sum_{i=1}^n \mu_k^i(A) - \sum_{i=1}^n \mu^i(A) \right| > \varepsilon \right) + \limsup_{k \rightarrow \infty} P \left( \left| \sum_{i=n}^{\infty} \mu_k^i(A) \right| > \varepsilon \right) \\ &\leq 0 + \limsup_{k \rightarrow \infty} \frac{E(|\sum_{i=n}^{\infty} \mu_k^i(A) - \sum_{i=n}^{\infty} \mu^i(A)|)}{\varepsilon} \leq \limsup_{k \rightarrow \infty} \frac{E(\sum_{i=n}^{\infty} \mu_k^i(A) + \sum_{i=n}^{\infty} \mu^i(A))}{\varepsilon} = \frac{2 \sum_{i=n}^{\infty} \nu^i(A)}{\varepsilon} \leq \eta \end{aligned}$$

Hence by taking limit  $\eta$  goes to 0 it follows that  $\lim_{k \rightarrow \infty} P(|\sum_{i=1}^{\infty} \mu_k^i(A) - \sum_{i=1}^{\infty} \mu^i(A)| > 2\varepsilon) = 0$ , i.e.  $\sum_{i=1}^{\infty} \mu_k^i(A)$  converges in probability to  $\sum_{i=1}^{\infty} \mu^i(A)$ .

We have that

$$E \left( \sum_{i=1}^{\infty} \mu_k^i(A) \right) = \sum_{i=1}^{\infty} \nu^i(A) = E \left( \sum_{i=1}^{\infty} \mu^i(A) \right)$$

and  $\sum_{i=1}^{\infty} \nu^i(A) = \nu(A) < \infty$ . Thus the statement follows from Lemma 3.6.  $\square$

**Theorem 9.2.** *Assume that  $\mathcal{C}_k(\tau)(X)$  converges in  $\mathcal{L}^1$  for every  $\tau$  on  $X_0$  with  $I_{\varphi}(\tau) < \infty$ . Let  $\nu$  be a finite Borel measure on  $X_0$  and  $A \subseteq X$  be a Borel set. Assume that  $\mathcal{C}_k(\nu_{\perp})(X)$  converges to 0 in probability. Then both  $\mathcal{C}_k(\nu)(A)$  and  $\mathcal{C}_k(\nu_R)(A)$  converges in probability to the same limit  $\mu(A)$  and  $E(\mu(A)) = \nu_R(A) \leq \nu(A)$ .*

*Proof.* Let us take a sequence  $(A_n)_{n \in \mathbb{N}}$  as in Proposition 5.1. We can further assume that  $(A_n)_{n \in \mathbb{N}}$  is a sequence of disjoint sets. Let  $\tau_n = \nu|_{A_n}$ . Then  $\nu = \nu_{\perp} + \sum \tau_n$ . We have that  $\mathcal{C}_k(\nu_R)(A)$  converges in  $\mathcal{L}^1$  by Proposition 9.1 to a limit  $\mu(A)$ . Thus  $\mathcal{C}_k(\nu_R)(A)$  converges in probability to  $\mu(A)$  and  $E(\mu(A)) = \nu_R(A)$ . Since  $\mathcal{C}_k(\nu_{\perp})(A) \leq \mathcal{C}_k(\nu_{\perp})(X)$  we have that  $\mathcal{C}_k(\nu_{\perp})(A)$  converges to 0 in probability. Thus  $\mathcal{C}_k(\nu)(A)$  converges in probability to  $\mu(A)$ .  $\square$

**Theorem 9.3.** *Assume that  $\mathcal{C}_k(\tau)(X)$  converges in  $\mathcal{L}^1$  for every  $\tau$  on  $X_0$  with  $I_{\varphi}(\tau) < \infty$ . Let  $\nu$  be a finite Borel measure on  $X$  such that  $\nu(X \setminus X_0) = 0$ . Assume that  $\mathcal{C}_k(\nu_{\perp})(X)$  converges to 0 in probability. Then  $\mu_k = \mathcal{C}_k(\nu)$  weakly converges to a random, finite, Borel measure  $\mathcal{C}(\nu)$  subsequentially in probability with  $\text{supp} \mathcal{C}(\nu) \subseteq \text{supp} \nu \cap B$  almost surely. Furthermore,  $\int_X f(x) d\mu_k(x)$  converges to a random variable  $S(f)$  in probability with  $E(|S(f)|) \leq \int_X |f(x)| d\nu(x)$  for every  $f : X \rightarrow \mathbb{R}$  Borel measurable function such that  $\int_X |f(x)| d\nu(x) < \infty$ . For every countable collection of bounded Borel measurable functions  $f_n : X \rightarrow \mathbb{R}$  we have that  $\int_X f_n(x) d\mathcal{C}(\nu)(x) = S(f_n)$  almost surely. In particular, for a countable collection of deterministic Borel sets  $A_n \subseteq X$  we have that  $S(\chi_{A_n}) = \mathcal{C}(\nu)(A_n)$  almost surely. We have that  $E(\mathcal{C}(\nu)(A)) = E(S(\chi_A)) = \nu_R(A) \leq \nu(A)$  for every Borel set  $A \subseteq X$ .*

*Proof.* By Theorem 9.2 the assumptions of Theorem 4.17 are satisfied. Hence the statement follows from Theorem 4.17, Remark 4.18 and Proposition 4.20.  $\square$

**Definition 9.4.** Let  $\nu$  be a finite locally finite, Borel measure on  $X$ . If  $\mathcal{C}(\nu)$  is a random, locally finite, Borel measure that satisfies the following:

- i.)  $\mathcal{C}_k(\nu)$  vaguely converges to a random, locally finite, Borel measure  $\mathcal{C}(\nu)$  in probability,
- ii.)  $\int_X f(x) d\mu_k(x)$  converges to a random variable  $S(f)$  in probability with  $E(|S(f)|) \leq \int_X |f(x)| d\nu(x)$  for every  $f : X \rightarrow \mathbb{R}$  Borel measurable function such that  $\int_X |f(x)| d\nu(x) < \infty$ ,
- iii.) for every countable collection of deterministic, bounded, compactly supported, Borel measurable functions  $f_n : X \rightarrow \mathbb{R}$  we have that  $\int_X f_n(x) d\mathcal{C}(\nu)(x) = S(f_n)$  almost surely,
- iv.) for every countable collection of deterministic, Borel sets  $A_n \subseteq X$  such that  $\overline{A_n}$  is compact we have that  $S(\chi_{A_n}) = \mathcal{C}(\nu)(A_n)$  almost surely,
- v.)  $E(\mathcal{C}(\nu)(A)) = E(S(\chi_A)) = \nu_R(A) \leq \nu(A)$  for every Borel set  $A \subseteq X$  with  $\nu(A) < \infty$ ,
- vi.) if  $A \subseteq X$  is a Borel set such that  $I_\varphi(\nu|_A) < \infty$  then  $\mathcal{C}_k(\nu)(A)$  converges to  $\mathcal{C}(\nu)(A)$  in  $\mathcal{L}^2$ ,
- vii.)  $\mathcal{C}_k(\nu_R)(A)$  converges to  $\mathcal{C}(\nu_R)(A) = \mathcal{C}(\nu)(A)$  in  $\mathcal{L}^1$  for every Borel set  $A \subseteq X$  with  $\nu(A) < \infty$
- viii.)  $\mathcal{C}(\nu_\perp) = 0$  almost surely,
- ix.)  $\mathcal{C}(\nu) = \mathcal{C}(\nu_R)$  almost surely,
- x.) if  $\nu = \sum_{i=1}^\infty \nu^i$  for a sequence of Borel measures  $\nu^i$  then  $\mathcal{C}(\nu) = \sum_{i=1}^\infty \mathcal{C}(\nu^i)$  almost surely,
- xi.) if  $\gamma \in [0, \infty)$  then  $\mathcal{C}(\gamma \cdot \nu) = \gamma \cdot \mathcal{C}(\nu)$  almost surely,
- xii.)  $\text{supp}\mathcal{C}(\nu) \subseteq \text{supp}\nu \cap B$  almost surely,

then we say that the *conditional measure of  $\nu$  on  $B$*  exists with respect to  $\mathcal{Q}_k$  ( $k \geq 1$ ) and it is  $\mathcal{C}(\nu)$ .

**Theorem 9.5.** Assume that  $B$  is almost surely a closed set and if  $C_\varphi(F) = 0$  for some compact set  $F \subseteq X$  then  $B \cap F = \emptyset$  almost surely. Assume that (15) holds for every  $Q \in \mathcal{Q}_k$ ,  $k \geq 1$  and that there exists  $0 < \delta < 1$  such that (16). Let  $\nu$  be a finite, Borel measure on  $X$  such that  $\nu(X \setminus X_0) = 0$ . Assume that at least one of the following conditions hold:

A.)  $\underline{E}(x, y) = \overline{F}(x, y)$  for  $\nu \times \nu$  almost every  $(x, y)$ ,

B.) for every  $A \subseteq X$  there exists a filtration  $\mathcal{F}_k$  such that  $\mathcal{C}_k(\nu)(A)$  is a martingale with respect to the filtration  $\mathcal{F}_k$ .

Then the conditional measure  $\mathcal{C}(\nu)$  of  $\nu$  on  $B$  exists with respect to  $\mathcal{Q}_k$  ( $k \geq 1$ ) and furthermore the following hold:

i\*)  $\mathcal{C}_k(\nu)$  weakly converges to a random, finite, Borel measure  $\mathcal{C}(\nu)$  subsequentially in probability, and hence weakly in probability and vaguely in probability by Proposition 4.23 and Remark 4.29,

iii\*) for every countable collection of bounded, deterministic, Borel measurable functions  $f_n : X \rightarrow \mathbb{R}$  we have that  $\int_X f_n(x) d\mathcal{C}(\nu)(x) = S(f_n)$  almost surely,

*iv\*.) for every countable collection of deterministic, Borel sets  $A_n \subseteq X$  we have that  $S(\chi_{A_n}) = \mathcal{C}(\nu)(A_n)$  almost surely.*

*Proof.* By Proposition 5.1 we have that  $\nu = \nu_\perp + \nu_R = \nu_\perp + \sum_{i \in \mathbb{N}} \nu|_{A_i}$  such that  $I_\varphi(\nu|_{A_i}) < \infty$  for every  $i \in \mathbb{N}$ . Then by Theorem 8.12 and Theorem 8.13 if  $A \subseteq X$  is a Borel set such that  $I_\varphi(\nu|_A) < \infty$  then  $\mathcal{C}_k(\nu)(A)$  converges in  $\mathcal{L}^2$  to a random variable  $\mu(A)$ . Note that if *iv\*.)* holds then *vi.)* holds. It follows that  $\mathcal{C}_k(\nu)(A)$  converges in  $\mathcal{L}^1$  to a random variable  $\mu(A) = \sum_{i \in \mathbb{N}} \mathcal{C}_k(\nu|_{A_i})(A)$  for every Borel set  $A \subseteq X$  with  $\nu_\perp(A) = 0$  by Proposition 9.1. By 6.4 we have that  $\mathcal{C}_k(\nu_\perp)(X)$  converges to 0 in probability. Thus by Theorem 9.3 we have that *i\*.), ii.), iii\*.), iv\*.), v.), xii.)* hold and so, as we noted, *vi.)* holds. From *vi.)* and Proposition 9.1 we have that *vii.)* holds. Clearly *viii.)* holds and so *ix.)* holds. Property *x.)* holds by Proposition 4.16 and *xi.)* holds trivially.  $\square$

**Theorem 9.6.** *Let  $X$  be locally compact. Assume that  $\mathcal{C}_k(\tau)(X)$  converges in  $\mathcal{L}^1$  for every  $\tau$  on  $X_0$  with  $I_\varphi(\tau) < \infty$ . Let  $\nu$  be a locally finite Borel measure on  $X$  such that  $\nu(X \setminus X_0) = 0$ . Assume that  $\mathcal{C}_k(\nu_\perp)(A)$  converges to 0 in probability for every compact set  $A$ . Then  $\mu_k = \mathcal{C}_k(\nu)$  vaguely converges to a random, finite, Borel measure  $\mathcal{C}(\nu)$  in probability with  $\text{supp} \mathcal{C}(\nu) \subseteq \text{supp} \nu \cap B$  almost surely. Furthermore,  $\int_X f(x) d\mu_k(x)$  converges to a random variable  $S(f)$  in probability with  $E(|S(f)|) \leq \int_X |f(x)| d\nu(x)$  for every  $f : X \rightarrow \mathbb{R}$  Borel measurable function such that  $\int_X |f(x)| d\nu(x) < \infty$ . For every countable collection of deterministic, compactly supported, bounded Borel measurable functions  $f_n : X \rightarrow \mathbb{R}$  we have that  $\int_X f_n(x) d\mathcal{C}(\nu)(x) = S(f_n)$  almost surely. In particular, for a countable collection of deterministic Borel sets  $A_n \subseteq X$  we have that  $S(\chi_{A_n}) = \mathcal{C}(\nu)(A_n)$  almost surely. We have that  $E(\mathcal{C}(\nu)(A)) = E(S(\chi_A)) = \nu_R(A) \leq \nu(A)$  for every Borel set  $A \subseteq X$  with  $\nu(A) < \infty$ .*

*Proof.* By Theorem 9.2 and Proposition 6.4 the assumptions of Theorem 4.37 are satisfied. Hence the statement follows from Theorem 4.37, Remark 4.38 and Proposition 4.40.  $\square$

**Theorem 9.7.** *Let  $X$  be locally compact. Assume that  $B$  is almost surely a closed set and if  $C_\varphi(F) = 0$  for some compact set  $F \subseteq X$  then  $B \cap F = \emptyset$  almost surely. Assume that (15) holds for every  $Q \in \mathcal{Q}_k$ ,  $k \geq 1$  and that there exists  $0 < \delta < 1$  such that (16). Let  $\nu$  be a locally finite, Borel measure on  $X$  such that  $\nu(X \setminus X_0) = 0$ . Assume that at least one of the following conditions hold:*

*A.)  $\underline{F}(x, y) = \overline{F}(x, y)$  for  $\nu \times \nu$  almost every  $(x, y)$ ,*

*B.) for every  $A \subseteq X$  there exists a filtration  $\mathcal{F}_k$  such that  $\mathcal{C}_k(\nu)(A)$  is a martingale with respect to the filtration  $\mathcal{F}_k$ .*

*Then the conditional measure  $\mathcal{C}(\nu)$  of  $\nu$  on  $B$  exists with respect to  $\mathcal{Q}_k$  ( $k \geq 1$ ).*

*Proof.* The assumptions of Theorem 9.6 are satisfied by Proposition 6.4 and Property *vi.)* which holds by Theorem 9.5. Then *i.), ii.), iii.), iv.), v.), xii.)* hold by Theorem 9.6. Property *vi.), vii.)* hold by Theorem 9.5. Property *viii.), ix.)* hold locally by Theorem 9.5 since  $X$  is locally compact and so hold globally since  $X$  is a separable metric space. Property *x.)* holds by Proposition 4.36 and *xi.)* holds trivially.  $\square$

## 10 Double integration

**Theorem 10.1.** *Assume that (15) holds and there exists  $0 < \delta < 1$  such that (16) and (13). Let  $\nu_1$  and  $\nu_2$  be finite Borel measures (continuous) on  $X$  with  $I_\varphi(\nu_1 + \nu_2) < \infty$ . Assume that the conditional measure  $\mathcal{C}_k(\nu_i)$  of  $\nu_i$  on  $B$  exists with respect to  $\mathcal{Q}_k$  ( $k \geq 1$ ) for  $i = 1, 2$ . Then*

$$\begin{aligned} \int_{A_2} \left( \int_{A_1} \underline{F}(x, y) d\nu_1(x) \right) d\nu_2(y) &\leq E(\mathcal{C}(\nu_1)(A_1) \cdot \mathcal{C}(\nu_2)(A_2)) \leq \int_{A_2} \left( \int_{A_1} \overline{F}(x, y) d\nu_1(x) \right) d\nu_2(y) \\ &\leq c \int_{A_2} \left( \int_{A_1} \varphi(x, y) d\nu_1(x) \right) d\nu_2(y) < \infty \end{aligned} \quad (27)$$

for every Borel sets  $A_1, A_2 \subseteq X$ .

*Proof.* By Property vi.) in Definition (9.4) it follows that  $\mathcal{C}_k(\nu_1)(A_1)$  converges to  $\mathcal{C}(\nu_1)(A_1)$  in  $\mathcal{L}^2$  and  $\mathcal{C}_k(\nu_2)(A_2)$  converges to  $\mathcal{C}(\nu_2)(A_2)$  in  $\mathcal{L}^2$ . Thus  $\mathcal{C}_k(\nu_1)(A_1) \cdot \mathcal{C}_k(\nu_2)(A_2)$  converges to  $\mathcal{C}(\nu_1)(A_1) \cdot \mathcal{C}(\nu_2)(A_2)$  in  $\mathcal{L}^1$  and in particular,

$$E(\mathcal{C}(\nu_1)(A_1) \cdot \mathcal{C}(\nu_2)(A_2)) = \lim_{k \rightarrow \infty} E(\mathcal{C}_k(\nu_1)(A_1) \cdot \mathcal{C}_k(\nu_2)(A_2)) = \lim_{k \rightarrow \infty} \int_{A_2} \left( \int_{A_1} F_{k,k}(x, y) d\nu_1(x) \right) d\nu_2(y) \quad (28)$$

by Lemma (8.3). We can assume that  $A_1$  and  $A_2$  are both  $X$  by restricting the measures to  $\nu_1|_{A_1}$  and  $\nu_2|_{A_2}$ .

Let  $\varepsilon, \eta > 0$  be fixed. Then, by Lemma (8.7) and by Lemma (8.9), for large enough  $k \in \mathbb{N}$

$$\begin{aligned} \int_X \int_X F_{k,k}(x, y) d\nu_1(x) d\nu_2(y) &\leq \eta + \iint_{A_\varepsilon} \overline{F}(x, y) d\nu_1(x) d\nu_2(y) + c_4 \iint_{H_\varepsilon} \varphi(x, y) d\nu(x) d\nu(y) + c_4 \cdot \nu \times \nu(H_\varepsilon) \\ &\leq \eta + \int_X \int_X \overline{F}(x, y) d\nu_1(x) d\nu_2(y) + c_4 \iint_{H_\varepsilon} \varphi(x, y) d\nu(x) d\nu(y) + c_4 \cdot \nu \times \nu(H_\varepsilon). \end{aligned} \quad (29)$$

Let  $D = \{(x, x) : x \in X\}$ . Since  $\int_X \int_X \varphi(x, y) d\nu(x) d\nu(y) < \infty$  and  $\nu(D) = 0$  by Fubini's theorem it follows that

$$\lim_{\varepsilon \rightarrow 0} \iint_{H_\varepsilon} \varphi(x, y) d\nu(x) d\nu(y) = 0 \quad (30)$$

by Remark (8.8). Similarly

$$\lim_{\varepsilon \rightarrow 0} \nu \times \nu(H_\varepsilon) = 0. \quad (31)$$

Hence by (28), (29), (30), (31) and Remark (8.6)

$$E(\mathcal{C}(\nu_1)(X) \cdot \mathcal{C}(\nu_2)(X)) \leq \int_X \int_X \overline{F}(x, y) d\nu_1(x) d\nu_2(y) \leq c \int_X \int_X \varphi(x, y) d\nu_1(x) d\nu_2(y) < \infty. \quad (32)$$

By Lemma (8.7), for large enough  $k \in \mathbb{N}$

$$-\eta + \iint_{A_\varepsilon} \underline{F}(x, y) d\nu_1(x) d\nu_2(y) \leq \iint_{A_\varepsilon} F_{k,k}(x, y) d\nu_1(x) d\nu_2(y) \leq \int_X \int_X F_{k,k}(x, y) d\nu_1(x) d\nu_2(y). \quad (33)$$

By Remark (8.6) we have that  $\underline{F}(x, y) \leq c \cdot \varphi(x, y)$  for  $\nu \times \nu$  almost every  $(x, y)$ . Thus similarly to (30)

$$\lim_{\varepsilon \rightarrow 0} \iint_{A_\varepsilon} \underline{F}(x, y) d\nu_1(x) d\nu_2(y) = \int_X \int_X \underline{F}(x, y) d\nu_1(x) d\nu_2(y). \quad (34)$$

Then it follows from (28), (33) and (34) that

$$\int_X \int_X \underline{F}(x, y) d\nu_1(x) d\nu_2(y) \leq E(\mathcal{C}(\nu_1)(A_1) \cdot \mathcal{C}(\nu_2)(A_2)). \quad (35)$$

So the statement follows from (32) and (35).  $\square$

**Theorem 10.2.** *Assume that (15) holds and there exists  $0 < \delta < 1$  such that (16) and (13). Let  $\nu_1$  and  $\nu_2$  be finite Borel measures (continuous) on  $X$  with  $I_\varphi(\nu_1 + \nu_2) < \infty$ . Assume that the conditional measure  $\mathcal{C}_k(\nu_i)$  of  $\nu_i$  on  $B$  exists with respect to  $\mathcal{Q}_k$  ( $k \geq 1$ ) for  $i = 1, 2$ . Let  $f(x, y) : X \times X \rightarrow \mathbb{R}$  be a nonnegative Borel function. Then*

$$\int \int \underline{F}(x, y) f(x, y) d\nu_1(x) d\nu_2(y) \leq E \left( \int \int f(x, y) d\mathcal{C}(\nu_1)(x) d\mathcal{C}(\nu_2)(y) \right) \leq \int \int \overline{F}(x, y) f(x, y) d\nu_1(x) d\nu_2(y) \quad (36)$$

*Proof.* It follows from Theorem (10.1) that (36) holds for functions of the form  $f(x, y) = I_{A_1}(x) \cdot I_{A_2}(y)$  for Borel sets  $A_1, A_2 \subseteq X$ . Hence, by the fact that the sets of the form  $Q \times S$  form a semi-ring generating the Borel  $\sigma$ -algebra of  $X \times X$  we can deduce that (36) holds for  $f(x, y) = I_A(x, y)$  for Borel sets  $A \subseteq X \times X$ . It follows that (36) holds for non-negative simple functions on  $X \times X$  and so we can deduce (36) for every nonnegative Borel function on  $X \times X$  using the monotone convergence theorem.  $\square$

**Corollary 10.3.** *Assume that (15) holds and there exists  $0 < \delta < 1$  such that (16) and (13). Let  $\nu_1$  and  $\nu_2$  be finite Borel measures (continuous) on  $X$  with  $I_\varphi(\nu_1 + \nu_2) < \infty$ . Assume that the conditional measure  $\mathcal{C}_k(\nu_i)$  of  $\nu_i$  on  $B$  exists with respect to  $\mathcal{Q}_k$  ( $k \geq 1$ ) for  $i = 1, 2$ . If  $F(x, y) = \underline{F}(x, y) = \overline{F}(x, y)$  for  $\nu \times \nu$  almost every  $(x, y)$  then*

$$E \left( \int \int f(x, y) d\mathcal{C}(\nu_1)(x) d\mathcal{C}(\nu_2)(y) \right) = \int \int F(x, y) f(x, y) d\nu_1(x) d\nu_2(y)$$

for every  $f(x, y) : X \times X \rightarrow \mathbb{R}$  Borel function with  $\int \int F(x, y) |f(x, y)| d\nu_1(x) d\nu_2(y) < \infty$ , in particular if  $\int_X \int_X \varphi(x, y) |f(x, y)| d\nu_1(x) d\nu_2(y) < \infty$ .

*Proof.* By Remark (8.6)  $\int_X \int_X \varphi(x, y) |f(x, y)| d\nu_1(x) d\nu_2(y) < \infty$  implies  $\int \int F(x, y) |f(x, y)| d\nu_1(x) d\nu_2(y) < \infty$ . The statement follows by applying Theorem (10.2) to  $f^+$  and  $f^-$ .  $\square$



**Corollary 10.4.** Assume that (15) holds and there exists  $0 < \delta < 1$  such that (16) and (13). Let  $\nu_1$  and  $\nu_2$  be finite Borel measures (continuous) on  $X$  with  $I_\varphi(\nu_1 + \nu_2) < \infty$ . Assume that the conditional measure  $\mathcal{C}_k(\nu_i)$  of  $\nu_i$  on  $B$  exists with respect to  $\mathcal{Q}_k$  ( $k \geq 1$ ) for  $i = 1, 2$ . Then

$$\lim_{k \rightarrow \infty} E(\mathcal{C}_k(\nu_1)(X)\mathcal{C}_k(\nu_2)(X)) = E(\mathcal{C}(\nu_1)(X)\mathcal{C}(\nu_2)(X)) \leq c \int \int \varphi(x, y) d\nu_1(x) d\nu_2(y).$$

*Proof.* By Theorem 10.2 and Remark (8.6) it follows that  $E(\mathcal{C}(\nu_1)(X)\mathcal{C}(\nu_2)(X)) \leq c \int \int \varphi(x, y) d\nu_1(x) d\nu_2(y)$ . By Property *vi.* in Definition (9.4) it follows that  $\mathcal{C}_k(\nu_1)(X)$  converges to  $\mathcal{C}(\nu_1)(X)$  in  $\mathcal{L}^2$  and  $\mathcal{C}_k(\nu_2)(X)$  converges to  $\mathcal{C}(\nu_2)(X)$  in  $\mathcal{L}^2$ . Thus  $\mathcal{C}_k(\nu_1)(X)\mathcal{C}_k(\nu_2)(X)$  converges to  $\mathcal{C}(\nu_1)(X)\mathcal{C}(\nu_2)(X)$  in  $\mathcal{L}^1$  and so the statement follows.  $\square$

**Theorem 10.5.** Assume that (15) holds and there exists  $0 < \delta < 1$  such that (16) and (13). Assume that whenever  $\nu_1$  and  $\nu_2$  are finite Borel measures on  $X$  with  $I_\varphi(\nu_i) < \infty$  for  $i = 1, 2$  then  $I_\varphi(\nu_1 + \nu_2) < \infty$ . Let either  $\nu$  and  $\tau$  be finite Borel measures or  $X$  be locally compact and  $\nu$  and  $\tau$  be locally finite Borel measures. Assume that the conditional measure  $\mathcal{C}_k(\nu)$  of  $\nu$  and  $\mathcal{C}_k(\tau)$  of  $\tau$  on  $B$  exists with respect to  $\mathcal{Q}_k$  ( $k \geq 1$ ). Let  $f(x, y) : X \times X \rightarrow \mathbb{R}$  be a nonnegative Borel function. Then

$$\int \int \underline{F}(x, y) f(x, y) d\nu_R(x) d\tau_R(y) \leq E \left( \int \int f(x, y) d\mathcal{C}(\nu)(x) d\mathcal{C}(\tau)(y) \right) \leq \int \int \overline{F}(x, y) f(x, y) d\nu_R(x) d\tau_R(y) \quad (37)$$

*Proof.* By Proposition 5.1 there exists a sequence of disjoint Borel sets  $A_{i,\nu} \subseteq X$  ( $i \in \mathbb{N}$ ) such that  $I_\varphi(\nu|_{A_{i,\nu}}) < \infty$  for every  $i \in \mathbb{N}$  and  $\nu = \nu_\perp + \sum_{i \in \mathbb{N}} \nu|_{A_{i,\nu}}$  and similarly  $A_{i,\tau} \subseteq X$  ( $i \in \mathbb{N}$ ) such that  $I_\varphi(\tau|_{A_{i,\tau}}) < \infty$  for every  $i \in \mathbb{N}$  and  $\tau = \tau_\perp + \sum_{i \in \mathbb{N}} \tau|_{A_{i,\tau}}$ . We let  $\nu_i = \nu|_{A_{i,\nu}}$  and  $\tau_i = \tau|_{A_{i,\tau}}$  for every  $i \in \mathbb{N}$ . Thus by assumption  $I_\varphi(\nu_i + \tau_j) < \infty$  for  $i, j \in \mathbb{N}$ . By Theorem 10.2

$$\int \int \underline{F}(x, y) f(x, y) d\nu_i(x) d\tau_j(y) \leq E \left( \int \int f(x, y) d\mathcal{C}(\nu_i)(x) d\mathcal{C}(\tau_j)(y) \right) \leq \int \int \overline{F}(x, y) f(x, y) d\nu_i(x) d\tau_j(y)$$

for every  $i, j \in \mathbb{N}$ . By summing over all  $i, j \in \mathbb{N}$  the statement follows by Property *viii.* in Definition 9.4.  $\square$

**Corollary 10.6.** Assume that (15) holds and there exists  $0 < \delta < 1$  such that (16) and (13). Assume that whenever  $\nu_1$  and  $\nu_2$  are finite Borel measures on  $X$  with  $I_\varphi(\nu_i) < \infty$  for  $i = 1, 2$  then  $I_\varphi(\nu_1 + \nu_2) < \infty$ . Let either  $\nu$  and  $\tau$  be finite Borel measures or  $X$  be locally compact and  $\nu$  and  $\tau$  be locally finite Borel measures. Assume that the conditional measure  $\mathcal{C}_k(\nu)$  of  $\nu$  and  $\mathcal{C}_k(\tau)$  of  $\tau$  on  $B$  exists with respect to  $\mathcal{Q}_k$  ( $k \geq 1$ ). If  $F(x, y) = \underline{F}(x, y) = \overline{F}(x, y)$  for  $\nu \times \nu$  almost every  $(x, y)$  then

$$E \left( \int \int f(x, y) d\mathcal{C}(\nu)(x) d\mathcal{C}(\tau)(y) \right) = \int \int F(x, y) f(x, y) d\nu_R(x) d\tau_R(y)$$

for every  $f(x, y) : X \times X \rightarrow \mathbb{R}$  Borel function with  $\int \int F(x, y) |f(x, y)| d\nu_R(x) d\tau_R(y) < \infty$ , in particular if  $\int_X \int_X \varphi(x, y) |f(x, y)| d\nu(x) d\tau(y) < \infty$ .

*Proof.* By Remark (8.6)  $\int_X \int_X \varphi(x, y) |f(x, y)| d\nu(x) d\tau(y) < \infty$  implies  $\int F(x, y) |f(x, y)| d\nu_R(x) d\tau_R(y) < \infty$ . The statement follows by applying Theorem (10.5) to  $f^+$  and  $f^-$ .  $\square$

## 11 Probability of non-extinction

**Definition 11.1.** The  $\alpha$ -capacity of  $\nu$  is

$$C_\varphi(\nu) = \sup \left\{ \frac{1}{I_\alpha(\tau)} : \tau \ll \nu, \tau(X) = 1 \right\}.$$

**Definition 11.2.** The upper  $\alpha$ -capacity of  $\nu$  is

$$\overline{C}_\varphi(\nu) = \inf \{ C_\varphi(A) : \nu(X \setminus A) = 0, A \subseteq X \text{ is Borel} \}.$$

**Theorem 11.3.** Assumptions what we need, then  $c^{-1} \cdot C_\varphi(\nu) \leq P(\mathcal{C}(\nu) \neq 0)$ .

*Proof.* For the lower bound let  $\tau \ll \nu$  with  $\tau(X) = 1$  and  $I_\alpha(\tau) < \infty$  (if there is no such  $\tau$  then the proof is trivial). Let  $A_N = \{x : \frac{d\tau}{d\nu}(x) \leq N\}$ . If  $D \subseteq A_N$  then

$$\tau(D) = \int_D \frac{d\tau}{d\nu}(x) d\nu(x) \leq \int_D N d\nu(x) = N\nu(D).$$

Hence  $\frac{1}{N}\mathcal{C}(\tau) \leq \mathcal{C}(\nu)$  on  $A_N$  and so  $\mathcal{C}(\nu) \equiv 0$  implies  $\mathcal{C}(\tau) \equiv 0$ . Thus

$$P(\mathcal{C}(\nu) \neq 0) \geq P(\mathcal{C}(\tau) \neq 0) \geq P\left(\bigcap_{n=1}^{\infty} \bigcup_{k \leq n} \{\mathcal{C}_k(\tau)(X) \geq \theta\}\right).$$

Using the continuity of probability measures we get that

$$P\left(\bigcap_{n=1}^{\infty} \bigcup_{k \leq n} \{\mathcal{C}_k(\tau)(X) \geq \theta\}\right) = \lim_n P\left(\bigcup_{k \leq n} \{\mathcal{C}_k(\tau)(X) \geq \theta\}\right) \geq \liminf P(\mathcal{C}_k(\tau)(X) \geq \theta).$$

Using Paley-Zygmund inequality and Proposition 10.4 it follows that

$$P(\mathcal{C}_k(\tau)(X) \geq \theta E(\mathcal{C}_k(\tau)(X))) \geq (1 - \theta)^2 \frac{E(\mathcal{C}_k(\tau)(X))^2}{E(\mathcal{C}_k(\tau)(X)^2)} \geq (1 - \theta)^2 \frac{1}{c \cdot I_\alpha(\tau)}.$$

Hence  $P(\mathcal{C}(\nu) \neq 0) \geq (1 - \theta)^2 \frac{1}{I_\alpha(\tau)}$  for every  $\theta > 0$  and  $\tau$ . Thus  $P(\mathcal{C}(\nu) \neq 0) \geq C_\varphi(\nu)$ .  $\square$

**Theorem 11.4.** Assume that  $B$  is almost surely a closed set and  $P(F \cap B \neq \emptyset) \leq b \cdot C_\varphi(F)$  for ever compact set  $F \subseteq X$ . If  $\nu$  is a finite Borel measure then  $P(\mathcal{C}(\nu) \neq 0) \leq b \cdot \overline{C}_\varphi(\nu)$ .

*Proof.* Let  $A_n \subseteq X$  be a sequence such that  $\nu(X \setminus A_n) = 0$  and  $C_\varphi(A_n) \leq \overline{C}_\varphi(\nu) + 1/n$ . Then for  $A := \bigcap_{n=1}^{\infty} A_n$  we have that  $\nu(X \setminus A) = 0$  and  $C_\varphi(A) = \overline{C}_\varphi(\nu)$ . Let  $F_n \subseteq A$  be an increasing sequence of compact sets such that  $\nu(X \setminus F_n) = \nu(A \setminus F_n) < 1/n$  and  $\lim_{n \rightarrow \infty} C_\varphi(F_n) = C_\varphi(A)$ . Whenever  $F_n \cap B = \emptyset$  then  $\mathcal{C}(\nu|_{F_n})(X) = 0$  by Lemma (6.3). Hence

$$\begin{aligned} P(\mathcal{C}(\nu|_{F_n})(X) > 0) &= P(\mathcal{C}(\nu|_{F_n})(X) > 0 \text{ and } F_n \cap B \neq \emptyset) \\ &\leq P(F_n \cap B \neq \emptyset) \leq b \cdot C_\varphi(F_n) \leq b \cdot C_\varphi(A) = b \cdot \overline{C}_\varphi(\nu). \end{aligned}$$

Let  $H_1 = F_1$  and  $H_n = F_n \setminus F_{n-1}$  for  $n \geq 2$ . Then  $\nu|_{F_n} = \sum_{i=1}^n \nu|_{H_i}$  and  $\nu = \sum_{i=1}^{\infty} \nu|_{H_i}$  since  $\nu(X \setminus F_n) = \nu(A \setminus F_n) < 1/n$ . Hence  $\mathcal{C}(\nu|_{F_n}) = \sum_{i=1}^n \mathcal{C}(\nu|_{H_i})$  and  $\mathcal{C}(\nu) = \sum_{i=1}^{\infty} \mathcal{C}(\nu|_{H_i})$  by Property  $x$ ) in Definition 9.4. Thus

$$P(\mathcal{C}(\nu)(X) > 0) = \lim_{n \rightarrow \infty} P(\mathcal{C}(\nu|_{F_n})(X) > 0) \leq b \cdot \overline{C}_\varphi(\nu).$$

$\square$

## 12 Scalar product of signed and complex measures of finite energy

In this section  $\varphi : X \times X \longrightarrow \mathbb{R} \cup \{\infty\}$  is a nonnegative Borel measurable function such that  $\varphi(x, y) \neq \infty$  for every  $x, y \in X$ ,  $x \neq y$ . For signed or complex Borel measures  $\mu$  and  $\nu$  let

$$I_\varphi(\mu, \nu) = \int_X \left( \int_X \varphi(x, y) d\mu(z) \right) d\bar{\nu}(y)$$

where  $\bar{\nu}$  denotes the conjugate of  $\nu$ . We denote by  $\mathcal{M}_+(X)$  the set of all finite Borel measures on  $X$ , by  $\mathcal{M}_1(X)$  the set of all Borel probability measures on  $X$ , by  $\mathcal{M}_l$  the set of all locally finite measures on  $X$ , by  $\mathcal{M}_s(X)$  the set of all signed Borel measures on  $X$  with finite total variation and by  $\mathcal{M}_c(X)$  the set of all complex Borel measures with finite total variation. Let  $\mathcal{M}_+^\varphi(X) = \{\nu \in \mathcal{M}_+(X) : |I_\varphi(\nu, \nu)| < \infty\}$ , let  $\mathcal{M}_{s,\varphi}(X) = \{\nu \in \mathcal{M}_s(X) : \nu = \mu - \tau, \mu, \tau \in \mathcal{M}_+^\varphi(X)\}$ ,  $\mathcal{M}_s^\varphi(X) = \{\nu \in \mathcal{M}_s(X) : |I_\varphi(\nu, \nu)| < \infty\}$ ,  $\mathcal{M}_{c,\varphi}(X) = \{\nu \in \mathcal{M}_c(X) : \nu = \mu + i\tau, \mu, \tau \in \mathcal{M}_s^\varphi(X)\}$  and  $\mathcal{M}_c^\varphi(X) = \{\nu \in \mathcal{M}_c(X) : |I_\varphi(\nu, \nu)| < \infty\}$ , etc. If we say some quantity  $a \geq 0$  we mean that either  $a \in \mathbb{R}$  is a nonnegative value or  $a = +\infty$ , what it means integral is infity, define. If we say that  $|I_\varphi(\mu, \nu)| < \infty$  then we mean that  $\varphi$  is  $\mu \times \nu$  integrable, i.e.  $\int_X \left( \int_X \varphi(x, y) d\tau_\mu(z) \right) d\tau_\nu(y) < \infty$  where  $\tau_\mu$  is the total variation of  $\mu$  and  $\tau_\nu$  is the total variation of  $\nu$ .

**Lemma 12.1.** *If  $I_\varphi(\nu, \nu) \geq 0$  for every  $\nu \in \mathcal{M}_{s,\varphi}(X)$  then  $\mathcal{M}_s^\varphi(X) = \mathcal{M}_{s,\varphi}(X)$  is a vector space over  $\mathbb{R}$ ,  $\mathcal{M}_c^\varphi(X) = \mathcal{M}_{c,\varphi}(X)$  is a vector space over  $\mathbb{C}$  and  $|I_\varphi(\mu, \nu)| < \infty$  for every  $\mu, \nu \in \mathcal{M}_c^\varphi(X)$ .*

*Proof.* Let  $\mu, \nu \in \mathcal{M}_+^\varphi(X)$ . Then

$$0 \leq I_\varphi(\mu - \nu, \mu - \nu) = I_\varphi(\mu, \mu) + I_\varphi(\nu, \nu) - I_\varphi(\mu, \nu) - I_\varphi(\nu, \mu)$$

hence by  $I_\varphi(\mu, \mu) < \infty$  and  $I_\varphi(\nu, \nu) < \infty$  it follows that  $I_\varphi(\mu, \nu) < \infty$  and  $I_\varphi(\nu, \mu) < \infty$ . Thus it follows that

$$\left| I_\varphi\left(\sum_{i=1}^n \alpha_i \mu_i, \sum_{i=1}^m \beta_i \mu_i\right) \right| < \infty$$

for  $\nu_i, \mu_j \in \mathcal{M}_+^\varphi(X)$ ,  $\alpha_i, \beta_j \in \mathbb{C}$ ,  $m, n \in \mathbb{N}$ ,  $i = 1, \dots, n$ ,  $j = 1, \dots, m$ . Hence the statement follows.  $\square$

**Lemma 12.2.** *If  $\varphi(x, y) = \varphi(y, x)$  for every  $x, y \in X$  and  $I_\varphi(\nu, \nu) \geq 0$  for every  $\nu \in \mathcal{M}_{s,\varphi}(X)$  then  $I_\varphi(\nu, \nu) \geq 0$  for every  $\nu \in \mathcal{M}_c^\varphi(X)$ . If additionally,  $I_\varphi(\nu, \nu) = 0$  for  $\nu \in \mathcal{M}_s^\varphi(X)$  if and only if  $\nu = 0$  then same holds for  $\nu \in \mathcal{M}_c^\varphi(X)$ .*

*Proof.* It follows from  $\varphi(x, y) = \varphi(y, x)$  that  $I_\varphi(\mu, \nu) = \overline{I_\varphi(\nu, \mu)}$  for  $\mu, \nu \in \mathcal{M}_c^\varphi(X)$ , in particular  $I_\varphi(\mu, \nu) = I_\varphi(\nu, \mu)$  for  $\mu, \nu \in \mathcal{M}_s^\varphi(X)$ . Let  $\nu \in \mathcal{M}_c^\varphi(X)$  and let  $a, b \in \mathcal{M}_s^\varphi(X)$  such that  $\nu = a + ib$ . By Lemma 12.1 and by the fact that  $I_\varphi(a, b) = I_\varphi(b, a)$  it follows that

$$I_\varphi(\nu, \nu) = I_\varphi(a, a) + I_\varphi(b, b) + iI_\varphi(b, a) - iI_\varphi(a, b) = I_\varphi(a, a) + I_\varphi(b, b) \geq 0. \quad (38)$$

If additionally,  $I_\varphi(\nu, \nu) = 0$  for  $\nu \in \mathcal{M}_s^\varphi(X)$  if and only if  $\nu = 0$  then equality holds in (38) if and only if  $a = 0$  and  $b = 0$ .  $\square$

**Proposition 12.3.** *Assume that  $\varphi(x, y) = \varphi(y, x)$  for every  $x, y \in X$  and  $I_\varphi(\nu, \nu) \geq 0$  for every  $\nu \in \mathcal{M}_{s,\varphi}(X)$ . Then  $\mathcal{M}_c^\varphi(X)$  is a vector space over  $\mathbb{C}$  and*

- i.)  $I_\varphi : \mathcal{M}_c^\varphi(X) \times \mathcal{M}_c^\varphi(X) \longrightarrow \mathbb{R}$  is a sesquilinear function,
- ii.)  $I_\varphi(\mu, \nu) = \overline{I_\varphi(\nu, \mu)}$  for  $\mu, \nu \in \mathcal{M}_c^\varphi(X)$ ,
- iii.)  $I_\varphi$  is a positive semidefinite on  $\mathcal{M}_c^\varphi(X)$ .

**Proposition 12.4.** *Assume that  $\varphi(x, y) = \varphi(y, x)$  for every  $x, y \in X$  and  $I_\varphi(\nu, \nu) \geq 0$  for every  $\nu \in \mathcal{M}_{s,\varphi}(X)$ . Then  $(\mathcal{M}_c^\varphi(X), I_\varphi)$  is an inner product space over  $\mathbb{C}$ , i.e.*

- i.)  $I_\varphi : \mathcal{M}_c^\varphi(X) \times \mathcal{M}_c^\varphi(X) \longrightarrow \mathbb{R}$  is a sesquilinear function,
- ii.)  $I_\varphi(\mu, \nu) = \overline{I_\varphi(\nu, \mu)}$  for  $\mu, \nu \in \mathcal{M}_c^\varphi(X)$ ,
- iii.)  $I_\varphi$  is a positive definite on  $\mathcal{M}_c^\varphi(X)$ .

Proposition 12.3 and Proposition 12.4 follows from Lemma 12.1 and Lemma 12.2.

**Lemma 12.5.** *Assume that  $\varphi(x, y) = \varphi(y, x)$  for every  $x, y \in X$  and  $I_\varphi(\nu, \nu) \geq 0$  for every  $\nu \in \mathcal{M}_{s,\varphi}(X)$ . Then*

$$|I_\varphi(\mu, \nu)|^2 \leq I_\varphi(\mu, \mu) \cdot I_\varphi(\nu, \nu) \quad (39)$$

for every  $\mu, \nu \in \mathcal{M}_c^\varphi(X)$ . If additionally  $I_\varphi(\nu, \nu) = 0$  for  $\nu \in \mathcal{M}_s^\varphi(X)$  if and only if  $\nu = 0$  then equality holds in (39) if and only if either  $\nu = 0$  or  $\mu = \alpha\nu$  for some  $\alpha \in \mathbb{R}$ .

Lemma 12.5 follows from the Cauchy-Schwarz inequality, Proposition 12.3 and Proposition 12.4.

**Lemma 12.6.** *Let  $G(x, y)$  be the Green's function of a transient Brownian motion (Definitions). Then  $I_\varphi(\nu, \nu) \geq 0$  for every  $\nu \in \mathcal{M}_{s,\varphi}(D)$ , equality holds if and only if  $\nu = 0$ .*

Lemma 12.6 can be find in [7, Lemma 8.29]. Then next corollary follows from Lemma 12.6 and Lemma 12.5.

**Corollary 12.7.** *Let  $G(x, y)$  be the Green's function of a transient Brownian motion (Definitions). Then  $(\mathcal{M}_c^G(D), I_G)$  is an inner product space over  $\mathbb{C}$ .*

**Lemma 12.8.** *Let the  $G(x, y)$  be the Green's function of a transient Brownian motion in  $\mathbb{R}^d$  for some  $d \geq 3$ . Then  $G(x, y) = c(d) \cdot \|x - y\|^{2-d}$  for  $x, y \in \mathbb{R}^d$ , where  $c(d)$  is a positive constant that depends only on  $d$ .*

Lemma 12.8 is shown for instance in [7, Theorem 3.33].

**Corollary 12.9.** *For some  $n = 1, 2, \dots$  let  $\varphi(x, y) = \|x - y\|^{-n}$  for  $x, y \in \mathbb{R}^d$ . Then  $(\mathcal{M}_c^\varphi(\mathbb{R}^d), I_\varphi)$  is an inner product space over  $\mathbb{C}$  for  $d = n + 1$  and  $d = n + 2$ .*

Corollary 12.9 follows from Lemma 12.8 and Corollary 12.7 by taking  $G(x, y) = c(n + 2)\varphi(x, y)$  to be the Green's function of a transient Brownian motion in  $\mathbb{R}^{n+2}$ .

**Lemma 12.10.** Let  $\varphi_0 = \max \{-\log \|x - y\|, 0\}$  for  $x, y \in \mathbb{R}^2$  and let  $G(x, y)$  be the Green's function of a transient Brownian motion in  $\mathbb{R}^2$ . Then there exists constants  $\alpha, \beta \geq 0$  such that

$$\frac{1}{\pi}\varphi_0(x, y) - \alpha \leq G(x, y) \leq \frac{1}{\pi}\varphi_0(x, y) + \beta$$

for every  $x, y \in \mathbb{R}^2$ .

Lemma 12.10 is shown in the proof of [7, Theorem 3.34].

**Corollary 12.11.** Let  $\varphi_0 = \max \{-\log \|x - y\|, 0\}$  for  $x, y \in \mathbb{R}^2$ . Then  $\mathcal{M}_c^{\varphi_0}(\mathbb{R}^2)$  is a vector space and  $|I_{\varphi_0}(\nu, \mu)| < \infty$  for every  $\mu, \nu \in \mathcal{M}_c^{\varphi_0}(\mathbb{R}^2)$ .

Corollary 12.11 follows from Lemma 12.10 and Corollary 12.7.

**Lemma 12.12.** If

$$I_\varphi(a, b) + I_\varphi(b, a) \leq I_\varphi(a, a) + I_\varphi(b, b) \quad (40)$$

for every  $a, b \in \mathcal{M}_+^\varphi(X)$ . Then  $I_\varphi(\nu, \nu) \geq 0$  for every  $\nu \in \mathcal{M}_{s,\varphi}(X)$ .

*Proof.* Let  $\nu \in \mathcal{M}_{s,\varphi}(X)$  and  $a, b \in \mathcal{M}_+^\varphi(X)$  such that  $\nu = a - b$ . Then by 40 it follows that

$$0 \leq I_\varphi(a, b) + I_\varphi(b, a) \leq I_\varphi(a, a) + I_\varphi(b, b) < \infty.$$

Thus

$$I_\varphi(\nu, \nu) = I_\varphi(a, a) + I_\varphi(b, b) - I_\varphi(a, b) - I_\varphi(b, a) \geq 0.$$

□

*Remark 12.13.* If  $\varphi(x, y) = \varphi(y, x)$  for every  $x, y \in X$  and

$$I_\varphi(a, b) \leq \sqrt{I_\varphi(a, a) \cdot I_\varphi(b, b)} \quad (41)$$

for every  $\mu, \nu \in \mathcal{M}_+^\varphi(X)$  then by the inequality between the arithmetic and geometric mean it follows that

$$0 \leq 2I_\varphi(a, b) \leq 2\sqrt{I_\varphi(a, a) \cdot I_\varphi(b, b)} \leq I_\varphi(a, a) + I_\varphi(b, b) < \infty.$$

Hence by Lemma 12.12 it follows that  $I_\varphi(\nu, \nu) \geq 0$  for every  $\nu \in \mathcal{M}_{s,\varphi}(X)$ .

**Lemma 12.14.** If  $\varphi$  is a bounded positive definite function on  $X \times X$  then  $I_\varphi(\nu, \nu) \geq 0$  for every  $\nu \in \mathcal{M}_{s,\varphi}(X)$ . ( $X$  is a Polish space)

For details see [8, Corollary 2.4.8].

**Proposition 12.15.** Assume that  $\varphi$  is a bounded positive definite function and  $\varphi(x, y) = \varphi(y, x)$  for every  $x, y \in X$ . Then  $\mathcal{M}_c^\varphi(X)$  is a vector space over  $\mathbb{C}$  and  $I_\varphi$  is a positive semidefinite on  $\mathcal{M}_c^\varphi(X)$ .

The statement follows from Proposition 12.3 and Lemma 12.14.

**Theorem 12.16.** *Let  $f : [0, \infty) \rightarrow [0, \infty)$  a convex, continuous, monotone decreasing function and let  $\varphi(x, y) = f(|x - y|)$  for  $x, y \in \mathbb{R}$ . Then  $\varphi$  is a bounded positive definite function.*

Polya's theorem see for example [8, Theorem 3.9.11].

**Theorem 12.17.** *Let  $f : [0, \infty) \rightarrow [0, \infty)$  be a continuous function, that is infinitely many times differentiable in  $(0, \infty)$  and*

$$(-1)^n f^{(n)}(r) \geq 0$$

*for every  $n \in \mathbb{N}$  and  $r > 0$ . Let  $\varphi(x, y) = f(\|x - y\|^2)$  for every  $x, y \in \mathbb{R}^d$ . Then  $\varphi$  is a positive definite on  $\mathbb{R}^d \times \mathbb{R}^d$  for every  $d \in \mathbb{N}$ .*

See [8, Theorem 3.9.8].

**Lemma 12.18.** *Let  $\varphi : X \times X \rightarrow \mathbb{R} \cup \{\infty\}$  is a nonnegative Borel measurable function such that  $\varphi(x, y) \neq \infty$  for every  $x, y \in X$ ,  $x \neq y$ . Let  $\varphi_\varepsilon$  be a bounded positive definite function on  $X \times X$  for every  $\varepsilon > 0$  such that  $0 \leq \varphi_\varepsilon \leq \varphi$  on  $X \times X$  for every  $\varepsilon \geq 0$  and  $\varphi_\varepsilon(x, y) \nearrow \varphi(x, y)$  as  $\varepsilon$  approaches 0 for every  $x, y \in X$ . Then  $I_\varphi(\nu, \nu) \geq 0$  for every  $\nu \in \mathcal{M}_{s, \varphi}(X)$ .*

*Proof.* Let  $\nu \in \mathcal{M}_{s, \varphi}(X)$  and  $a, b \in \mathcal{M}_+^\varphi(X)$  such that  $\nu = a - b$ . Then  $\nu \in \mathcal{M}_{s, \varphi_\varepsilon}(X) = \mathcal{M}_s(X)$  since  $\varphi_\varepsilon$  is bounded. Thus

$$0 \leq I_{\varphi_\varepsilon}(a, b) + I_{\varphi_\varepsilon}(b, a) \leq I_{\varphi_\varepsilon}(a, a) + I_{\varphi_\varepsilon}(b, b) \leq I_\varphi(a, a) + I_\varphi(b, b) < \infty$$

by Lemma 12.14. Thus by the monotone convergence theorem it follows that

$$0 \leq I_\varphi(a, b) + I_\varphi(b, a) \leq I_\varphi(a, a) + I_\varphi(b, b) < \infty.$$

Then the statement follows by Lemma 12.14. □

**Lemma 12.19.** *Let  $\varphi : X \times X \rightarrow \mathbb{R} \cup \{\infty\}$  is a nonnegative Borel measurable function such that  $\varphi(x, y) \neq \infty$  for every  $x, y \in X$ ,  $x \neq y$ . Let  $\varphi_\varepsilon$  be a bounded positive definite function on  $X \times X$  for every  $\varepsilon > 0$  such that  $0 \leq \varphi_\varepsilon \leq \varphi$  on  $X \times X$  for every  $\varepsilon \geq 0$  and  $\varphi_\varepsilon(x, y) \nearrow \varphi(x, y)$  as  $\varepsilon$  approaches 0 for every  $x, y \in X$ . Then  $\mathcal{M}_c^\varphi(X)$  is a vecotor space over  $\mathbb{C}$  and  $I_\varphi$  is a positive semidefinite on  $\mathcal{M}_c^\varphi(X)$ .*

The statement follows from Proposition 12.18 and Lemma 12.14.

**Proposition 12.20.** *Let  $0 \leq \alpha < \infty$  and  $\varphi(x, y) = \|x - y\|^{-\alpha}$  for  $x, y \in \mathbb{R}^d$ . Then  $\mathcal{M}_c^\varphi(\mathbb{R}^d)$  is a vecotor space over  $\mathbb{C}$  and  $I_\varphi$  is a positive semidefinite on  $\mathcal{M}_c^\varphi(\mathbb{R}^d)$  for every  $d \in \mathbb{N}$ .*

*Proof.* Let  $f_\varepsilon(r) = (r + \varepsilon)^{-\alpha/2}$  for every  $r \in [0, \infty)$  and let  $\varphi_\varepsilon(x, y) = (\|x - y\| + \varepsilon)^{-\alpha}$  for every  $x, y \in \mathbb{R}^d$ . Then  $\varphi_\varepsilon$  is a bounded positive definite function on  $\mathbb{R}^d \times \mathbb{R}^d$  for every  $d \in \mathbb{N}$ . Thus the statement follows by Lemma 12.19. □

## 13 Brownian path

**Lemma 13.1.** *Let  $B$  be a Brownian path in  $\mathbb{R}^d$  for  $d \geq 3$ . Let  $x \in \mathbb{R}^d$ ,  $x \neq 0$  and  $r > 0$  such that  $\|x\| > r$ . Then*

$$P(B \cap B(x, r) \neq \emptyset) = \frac{r^{d-2}}{\|x\|^{d-2}}.$$

See [7, Corollary 3.19].

**Proposition 13.2.** *Let  $B$  be a Brownian path in  $\mathbb{R}^d$  for  $d \geq 3$ . Then for  $x, y \in \mathbb{R}^d \setminus \{0\}$ ,  $x \neq y$*

$$\begin{aligned} & \liminf_{R \rightarrow 0} \liminf_{r \rightarrow 0} \frac{P(B \cap B(x, R) \neq \emptyset, B \cap B(y, r) \neq \emptyset)}{P(B \cap B(x, R) \neq \emptyset) \cdot P(B \cap B(y, r) \neq \emptyset)} \\ &= \limsup_{R \rightarrow 0} \limsup_{r \rightarrow 0} \frac{P(B \cap B(x, R) \neq \emptyset, B \cap B(y, r) \neq \emptyset)}{P(B \cap B(x, R) \neq \emptyset) \cdot P(B \cap B(y, r) \neq \emptyset)} = \frac{\|x\|^{d-2} + \|y\|^{d-2}}{\|x - y\|^{d-2}}. \end{aligned} \quad (42)$$

*Proof.* Let  $W(t)$  be a Brownian motion in  $\mathbb{R}^d$  for some  $d \geq 3$ , so  $B = \{W(t) : t \in [0, \infty)\}$ . We denote by  $P = P_0$  the probability measure that corresponds to the Brownian motion that is started at the origin and by  $P_x$  the probability measure that corresponds to the Brownian motion that is started at  $x \in \mathbb{R}^d$ . Let

$$T_{x,r} = \inf \{t \in [0, \infty) : W(t) \in \partial B(x, r)\}$$

for  $x \in \mathbb{R}^d$  and  $r > 0$ . Let  $x, y \in \mathbb{R}^d$  and  $R, r > 0$  such that  $r + R < \|x - y\|$ ,  $R < \|x\|$  and  $r < \|y\|$ . If  $z \in \partial B(x, R)$  then

$$P_z(T_{y,r} < \infty) = \frac{r^{d-2}}{\|x - z\|^{d-2}}$$

thus

$$\frac{r^{d-2}}{(\|x - y\| + R)^{d-2}} \leq P_z(T_{y,r} < \infty) \leq \frac{r^{d-2}}{(\|x - y\| - R)^{d-2}} \quad (43)$$

and similarly for  $z \in \partial B(y, r)$

$$\frac{R^{d-2}}{(\|x - y\| + r)^{d-2}} \leq P_z(T_{x,R} < \infty) \leq \frac{R^{d-2}}{(\|x - y\| - r)^{d-2}}. \quad (44)$$

Let  $U = W(T_{x,R})$  and  $V = W(T_{y,r})$  be the stopped Brownian motions. Then by Lemma 13.1, (43) and by the Markov property

$$\begin{aligned} & P(W \text{ hits } B(x, R) \text{ and after that } W \text{ hits } B(y, r)) \\ &= P(T_{x,R} < \infty) \cdot E(P_U(T_{y,r} < \infty)) \leq \frac{R^{d-2}}{\|x\|^{d-2}} \cdot \frac{r^{d-2}}{(\|x - y\| - R)^{d-2}} \end{aligned}$$

and similarly

$$P(W \text{ hits } B(y, r) \text{ and after that } W \text{ hits } B(x, R))$$

$$= P(T_{y,r} < \infty) \cdot E(P_V(T_{x,R} < \infty)) \leq \frac{r^{d-2}}{\|y\|^{d-2}} \cdot \frac{R^{d-2}}{(\|x-y\| - r)^{d-2}}.$$

Hence

$$\begin{aligned} & P(B \cap B(x, R) \neq \emptyset \text{ and } B \cap B(y, r) \neq \emptyset) \\ & \leq \frac{R^{d-2}}{\|x\|^{d-2}} \cdot \frac{r^{d-2}}{(\|x-y\| - R)^{d-2}} + \frac{r^{d-2}}{\|y\|^{d-2}} \cdot \frac{R^{d-2}}{(\|x-y\| - r)^{d-2}} \\ & = r^{d-2} R^{d-2} \frac{\|y\|^{d-2} (\|x-y\| - r)^{d-2} + \|x\|^{d-2} (\|x-y\| - R)^{d-2}}{\|y\|^{d-2} (\|x-y\| - r)^{d-2} \cdot \|x\|^{d-2} (\|x-y\| - R)^{d-2}} \\ & \leq \frac{R^{d-2}}{\|x\|^{d-2}} \cdot \frac{r^{d-2}}{\|y\|^{d-2}} \cdot \frac{\|y\|^{d-2} \|x-y\|^{d-2} + \|x\|^{d-2} \|x-y\|^{d-2}}{(\|x-y\| - r)^{d-2} \cdot (\|x-y\| - R)^{d-2}}. \end{aligned} \quad (45)$$

It follows that

$$\begin{aligned} & \limsup_{R \rightarrow 0} \limsup_{r \rightarrow 0} \frac{P(B \cap B(x, R) \neq \emptyset, B \cap B(y, r) \neq \emptyset)}{P(B \cap B(x, R) \neq \emptyset) \cdot P(B \cap B(y, r) \neq \emptyset)} \\ & \leq \limsup_{R \rightarrow 0} \limsup_{r \rightarrow 0} \frac{\|y\|^{d-2} \|x-y\|^{d-2} + \|x\|^{d-2} \|x-y\|^{d-2}}{(\|x-y\| - r)^{d-2} \cdot (\|x-y\| - R)^{d-2}} = \frac{\|x\|^{d-2} + \|y\|^{d-2}}{\|x-y\|^{d-2}}. \end{aligned} \quad (46)$$

By (45) it follows that

$$\begin{aligned} & P(W \text{ hits } B(x, R) \text{ before } W \text{ hits } B(y, r)) \\ & \geq P(B \cap B(x, R) \neq \emptyset) - P(B \cap B(x, R) \neq \emptyset, B \cap B(y, r) \neq \emptyset) \geq \frac{R^{d-2}}{\|x\|^{d-2}} \cdot (1 - O(r)) \end{aligned} \quad (47)$$

and similarly

$$P(W \text{ hits } B(y, r) \text{ before } W \text{ hits } B(x, R)) \geq \frac{r^{d-2}}{\|y\|^{d-2}} \cdot (1 - O(R)). \quad (48)$$

Thus by Lemma 13.1, (43), (44) and by the Markov property

$$\begin{aligned} & P(B \cap B(x, R) \neq \emptyset, B \cap B(y, r) \neq \emptyset) \\ & \geq \frac{R^{d-2}}{\|x\|^{d-2}} \cdot (1 - O(r)) \cdot \frac{r^{d-2}}{(\|x-y\| + R)^{d-2}} + \frac{r^{d-2}}{\|y\|^{d-2}} \cdot (1 - O(R)) \cdot \frac{R^{d-2}}{(\|x-y\| + r)^{d-2}} \\ & = R^{d-2} r^{d-2} \frac{(1 - O(r)) \|y\|^{d-2} (\|x-y\| + r)^{d-2} + (1 - O(R)) \|x\|^{d-2} (\|x-y\| + R)^{d-2}}{\|y\|^{d-2} (\|x-y\| + r)^{d-2} \|x\|^{d-2} (\|x-y\| + R)^{d-2}} \\ & = \frac{R^{d-2}}{\|x\|^{d-2}} \cdot \frac{r^{d-2}}{\|y\|^{d-2}} \cdot \frac{(1 - O(r)) \|y\|^{d-2} (\|x-y\| + r)^{d-2} + (1 - O(R)) \|x\|^{d-2} (\|x-y\| + R)^{d-2}}{(\|x-y\| + r)^{d-2} (\|x-y\| + R)^{d-2}}. \end{aligned}$$

Hence

$$\liminf_{R \rightarrow 0} \liminf_{r \rightarrow 0} \frac{P(B \cap B(x, R) \neq \emptyset, B \cap B(y, r) \neq \emptyset)}{P(B \cap B(x, R) \neq \emptyset) \cdot P(B \cap B(y, r) \neq \emptyset)} \geq \frac{\|x\|^{d-2} + \|y\|^{d-2}}{\|x-y\|^{d-2}}. \quad (49)$$

So (42) holds by (46) and (49).  $\square$



**Lemma 13.3.** *Let  $B$  be a Brownian path in  $\mathbb{R}^d$  for  $d \geq 3$ , let  $0 \in A \subseteq \mathbb{R}^d$  be a compact set such that  $P(A \cap B) > 0$ . There exists a constant  $c_A > 0$ , depending on  $A$  such that for every  $x \in \mathbb{R}^d \setminus \{0\}$  and  $0 < r < 1/(2\text{diam}(A))$*

$$C_G(A) \frac{r^{d-2}}{\|x\|^{d-2}} (1 - r \cdot \text{diam}(A)) \leq P((r \cdot A + x) \cap B \neq \emptyset) \leq C_G(A) \frac{r^{d-2}}{\|x\|^{d-2}} (1 + 2r \cdot \text{diam}(A)).$$

*Proof.* Let  $G(x, y) = \|x - y\|^{2-d}$  be the Green's function of the Brownian motion, see Lemma 12.8. By [7, Corollary 8.12] and [7, Theorem 8.27]

$$C_G(r \cdot A + x) ((1 + r \cdot \text{diam}(A)) \|x\|)^{2-d} \leq P((r \cdot A + x) \cap B \neq \emptyset) \leq C_G(r \cdot A + x) ((1 - r \cdot \text{diam}(A)) \|x\|)^{2-d} \quad (50)$$

On the other hand by the scaling invariance of capacity it follows that

$$C_G(r \cdot A + x) = r^{d-2} C_G(A). \quad (51)$$

We have that  $1 - x \leq 1/(1 + x)$  and  $(1 - r \cdot \text{diam}(A))^{-1} \leq 1 + 2r \cdot \text{diam}(A)$  because  $0 < r < 1/(2\text{diam}(A))$ . Thus it follows from (50) and (51) that

$$C_G(A) \frac{r^{d-2}}{\|x\|^{d-2}} (1 - r \cdot \text{diam}(A))^{d-2} \leq P((r \cdot A + x) \cap B \neq \emptyset) \leq C_G(A) \frac{r^{d-2}}{\|x\|^{d-2}} (1 + 2r \cdot \text{diam}(A))^{d-2}$$

and so the statement follows for  $c_A = C_G(A)$ .  $\square$

**Proposition 13.4.** *Let  $B$  be a Brownian path in  $\mathbb{R}^d$  for  $d \geq 3$ , let  $0 \in A \subseteq \mathbb{R}^d$  be a compact set such that  $P(A \cap B) > 0$ . Then for  $x, y \in \mathbb{R}^d \setminus \{0\}$ ,  $x \neq y$*

$$\begin{aligned} & \liminf_{R \rightarrow 0} \liminf_{r \rightarrow 0} \frac{P((R \cdot A + x) \cap B \neq \emptyset, (r \cdot A + y) \cap B \neq \emptyset)}{P((R \cdot A + x) \cap B \neq \emptyset) \cdot P((r \cdot A + y) \cap B \neq \emptyset)} \\ &= \limsup_{R \rightarrow 0} \limsup_{r \rightarrow 0} \frac{P((R \cdot A + x) \cap B \neq \emptyset, (r \cdot A + y) \cap B \neq \emptyset)}{P((R \cdot A + x) \cap B \neq \emptyset) \cdot P((r \cdot A + y) \cap B \neq \emptyset)} = \frac{\|x\|^{d-2} + \|y\|^{d-2}}{\|x - y\|^{d-2}}. \end{aligned}$$

The proof of Proposition (13.4) goes similar to the proof of Proposition (13.2) replacing the use of Lemma (13.1) by Lemma (13.3).

Let  $Q_k(x)$  be the dyadic cube  $[\frac{i_1}{2^{-k}}, \frac{i_1+1}{2^{-k}}) \times \dots \times [\frac{i_d}{2^{-k}}, \frac{i_d+1}{2^{-k}})$  for  $i_1, \dots, i_d \in \mathbb{Z}$  such that  $x \in Q_k(x)$  and let  $\mathcal{Q}_k = \{Q_k(x) : x \in \mathbb{R}^d\}$ .

**Proposition 13.5.** *Let  $B$  be a Brownian path in  $\mathbb{R}^d$  for  $d \geq 3$ . Then*

$$\begin{aligned} & \liminf_{n \rightarrow \infty} \liminf_{k \rightarrow \infty} \frac{P(B \cap Q_n(x) \neq \emptyset, B \cap Q_k(y) \neq \emptyset)}{P(B \cap Q_n(x) \neq \emptyset) \cdot P(B \cap Q_k(y) \neq \emptyset)} \\ &= \limsup_{n \rightarrow \infty} \limsup_{k \rightarrow \infty} \frac{P(B \cap Q_n(x) \neq \emptyset, B \cap Q_k(y) \neq \emptyset)}{P(B \cap Q_n(x) \neq \emptyset) \cdot P(B \cap Q_k(y) \neq \emptyset)} = \frac{\|x\|^{d-2} + \|y\|^{d-2}}{\|x - y\|^{d-2}}. \end{aligned}$$

The proposition is a special case of Proposition (13.4).

*Proof of Theorem (1.1).* Let  $A(x, r) = \{(y_1, \dots, y_d) \in \mathbb{R}^d : |x_i - y_i| < r \text{ for every } i = 1, \dots, d\}$  and let  $A_{p,q} = A(0, 2^q) \setminus A(0, 2^{-p})$  for  $p, q \in \mathbb{N}$ . The conditions of Theorem (9.7) are satisfied for  $B$  and  $\widetilde{\mathcal{Q}}_k = \{Q \in \mathcal{Q}_k : Q \subseteq A_{p,q}\}$  by [7, Theorem 8.24], Proposition (13.5) and Proposition (12.7), furthermore

$$F = \frac{\|x\|^{d-2} + \|y\|^{d-2}}{\|x - y\|^{d-2}}.$$

Thus the statements hold by Theorem (9.7) and Corollary (10.6) for  $\nu|_{A_{p,q}}$  and so it follows by taking  $p$  and  $q$  go to  $\infty$ .

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